Basic Course on Stochastic Programming
List 1.

1. For the two-stage linear stochastic programming problem, assume the random vector ξ has finite support. Using the one-stage equivalent formulation deduce the optimality conditions for the two-stage problem.

2. For the two-stage linear stochastic programming problem, suppose the random vector ξ has finite support. Show that the problem has relatively complete recourse if and only if $Q(x,\xi) < +\infty$ for all feasible x and all ξ.

3. For the two-stage linear stochastic programming problem with fixed recourse. Show that the following statements are equivalent:

   (a) The problem has complete recourse.

   (b) The dual problem feasible set $\Pi(q)$ is bounded for all q.

   (c) The system $W^\top \pi \leq 0$ has a unique solution.

4. Show that if the dual problem of

$$\begin{array}{lll} v(z,\xi) := & \min_y & q_\xi^\top y \\ & \text{s.t.} & W_\xi y = z \\ & & y \geq 0 \end{array}$$

   has a unique solution, then $v(\cdot,\xi)$ is diffentiable at z. Find the gradient of $Q(x,\xi) = v(h_\xi - T_\xi x, \xi)$, when $z = h_\xi - T_\xi x$

5. Considering that the problem (parameterized by ξ)

$$\begin{array}{lll} \min_y & q_\xi^\top y \\ \text{s.t.} & W_\xi y = z \\ & y \geq 0 \end{array}$$

   has solution a.e $[\xi]$, show that there exists a Carathéodory function $c(\pi,\xi)$ such that the dual solution set $\mathcal{D}(\xi)$ can be expressed as

$$\mathcal{D}(\xi) = \{\pi : c(\pi,\xi) \leq 0\}$$

6. Consider a probability space Ω and a function $f : \mathbb{R}^n \times \Omega :\to \mathbb{R}$. Assume that

   (a) $\text{epi}(f(\cdot,\omega))$ is closed a. e.$[\xi]$

   (b) There exists a enumerable family $g_n : \Omega \to \mathbb{R}^n \times \mathbb{R}$, $n \in \mathbb{N}$, of measurable functions such that $\{g_n(\omega)\}_{n=1}^\infty$ is dense in $\text{epi}(f(\cdot,\omega))$, a.e $[\omega]$.

   For a subset $S \subset \mathbb{R}^n$, show that the function $F(\omega) = \inf_{x\in S} f(x,\omega)$ is measurable.

7. Consider the random vector $\xi : \Omega \to \mathbb{R}^n$. For a fixed vector $x \in \mathbb{R}^n$ prove that

$$\text{Var}[\xi^\top x] = x^\top \Sigma x,$$

where $\Sigma = \mathbb{E}[(\xi - \mu)^\top (\xi - \mu)]$ and $\mu = \mathbb{E}[\xi]$.

8. Consider the news vendor problem in which the news vendor buys x newspapers at a unit price c. Suppose $x \leq u$. The vendor sells the newspapers at a price q and unsold newspapers fetch a salvage value r. Suppose the demand for newspapers is denoted by a random variable ξ. Note that vendors cannot return to the publisher to buy more newspapers during the day.

   (a) Formulate the news vendor problem as a two-stage stochastic program and define the recourse function.

   (b) Suppose ξ is a continuous random variable, derive an expression for $Q(x, \xi)$. Comment on the nature of the recourse.

   (c) Derive the optimal number of newspapers x.