Risk Measures

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AVaR for General Distributions

Proposition

Given a convex function $f : \mathcal{M} \to \mathbb{R}$ that is positive homogeneous. The following statements are equivalent:

1. There exists $M > 0$ s. t. $f(x) \leq M$ for all $\|x\| \leq 0$.
2. There exists $M > 0$ s. t. $|f(x)| \leq M\|x\|$ for all x.
3. There exists $M > 0$ s. t. $|f(x) - f(y)| \leq M\|x - y\|$ for all x, y.

Corollary

AVaR is Lipschitz continuous.

Theorem

Any coherent risk measure that is everywhere finite is Lipschitz continuous and subdifferentiable.

AVaR on Optimization Problems

$$
\left[\begin{array}{rl} \min_x & c^\top x + \mathbb{E}[Q(x,\xi)] \\ \text{s. t.} & Ax = b \\ & x \geq 0 \\ Q(x,\xi) := & \begin{array}{ll} \min_y & q_\xi^\top y \\ \text{s. t.} & T_\xi x + W_\xi y = h_\xi \\ & y \geq 0 \end{array} \end{array}\right] \Leftrightarrow \left[\begin{array}{ll} \min_x & c^\top x + \mathbb{E}[q_\xi^\top y_\xi] \\ \text{s. t.} & Ax = b \\ & x \geq 0 \\ \text{s. t.} & T_\xi x + W_\xi y_\xi = h_\xi \\ & y_\xi \geq 0, \quad \forall \xi \end{array}\right]
$$

AVaR on Optimization Problems

$$\left[\begin{array}{lll} \min_x & c^\top x + AVaR_\alpha[Q(x,\xi)] & \\ \text{s. t.} & Ax = b & \\ & x \geq 0 & \\ Q(x,\xi) := & \min_y & q_\xi^\top y \\ & \text{s. t.} & T_\xi x + W_\xi y = h_\xi \\ & & y \geq 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{ll} \min_x & c^\top x + AVaR_\alpha[q_\xi^\top y_\xi] \\ \text{s. t.} & Ax = b \\ & x \geq 0 \\ \text{s. t.} & T_\xi x + W_\xi y_\xi = h_\xi \\ & y_\xi \geq 0, \quad \forall \xi \end{array}\right]$$

AVaR on Optimization Problems

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$$\Updownarrow$$

$$\left[\begin{array}{ll} \min_{x,u} & c^\top x + u + \frac{1}{\alpha}\mathbb{E}[[q_\xi^\top y_\xi - u]^+] \\ \text{s. t.} & Ax = b \\ & x \geq 0 \\ \text{s. t.} & T_\xi x + W_\xi y_\xi = h_\xi \\ & y_\xi \geq 0, \quad \forall \xi \end{array}\right]$$

AVaR on Optimization Problems

$$\left[\begin{array}{ll} \min_x & c^\top x + AVaR_\alpha[q_\xi^\top y_\xi] \\ \text{s. t.} & Ax = b \\ & x \geq 0 \\ \text{s. t.} & T_\xi x + W_\xi y_\xi = h_\xi \\ & y_\xi \geq 0, \quad \forall \xi \end{array}\right]$$

$$\Updownarrow$$

$$\left[\begin{array}{ll} \min_{x,u} & c^\top x + u + \frac{1}{\alpha}\mathbb{E}[t_\xi] \\ \text{s. t.} & Ax = b \\ & x \geq 0 \\ & q_\xi^\top y_\xi - u \leq t_\xi \\ & 0 \leq t_\xi \\ \text{s. t.} & T_\xi x + W_\xi y_\xi = h_\xi \\ & y_\xi \geq 0, \quad \forall \xi \end{array}\right]$$

AVaR on Optimization Problems

Constraints

$$AVaR_\alpha(Z) \le v$$

$$\Updownarrow$$

$$u + \frac{1}{\alpha}\mathbb{E}[[Z - u]^+] \le v$$

$$\Updownarrow$$

$$\begin{bmatrix} u + \dfrac{1}{\alpha}\displaystyle\sum_\xi p_\xi t_\xi \le v \\ Z_\xi - u \le t_\xi \\ 0 \le t_\xi \end{bmatrix}$$

Acceptance Sets

Given a space of random variables $\mathcal{M}$ (e.g. $L^p, 1 \leq p \leq +\infty$), denoting Considering a subset $\mathcal{A} \subset \mathcal{M}$ (called an acceptance set) we define a risk measure $\rho_{\mathcal{A}} : \mathcal{M} \to \overline{\mathbb{R}}$ by

$$\rho_{\mathcal{A}}(Z) := \inf\{t : Z - t \in \mathcal{A}\}$$

" The minimum amount of money we should pay to make some cost acceptable".

Remark. $\rho_{\mathcal{A}}$ is translation equivariant.

Acceptance Sets

Consider $L_{-} = \{Z \in \mathcal{M} : Z \leq 0(w.p.1)\}$

Proposition

Assume that

- $L_{-} \subset \mathcal{A}$.
- $\mathcal{A}$ is a cone (i.e. $tZ \in \mathcal{A}$, $\forall Z \in \mathcal{A}$, $\forall t \geq 0$)
- $\mathcal{A}$ is convex (i.e. $Y + Z \in \mathcal{A}$, $\forall Y, Z \in \mathcal{A}$)
- $\mathcal{A} \cap \mathbb{R}_{++} = \phi$.

Then, $\rho_{\mathcal{A}}$ is coherent. Moreover, if $\mathcal{A}$ is closed, then $\rho_{\mathcal{A}}$ is l.s.c.

Example. Consider $\mathcal{A} = L_{-}$

Acceptance Sets

Proposition

Consider a coherent risk measure $\rho : \mathcal{M} \to \overline{\mathbb{R}}$ and $\mathcal{A} := \{Z \in \mathcal{M} : \rho(Z) \leq 0\}$. We have that

1. $L_- \subset \mathcal{A}$
2. $\mathcal{A}$ is a convex cone.
3. if ρ is l.s.c., then $\mathcal{A}$ is closed.
4. $\mathcal{A} \cap \mathbb{R}_{++} = \phi$.
5. $\rho = \rho_{\mathcal{A}}$.