Given a random variable $Z$.

1. For $0 < \alpha < 1$ we have that
   \[ \mathbb{P}[Z > \text{VaR}_\alpha(Z)] \leq \alpha \leq \mathbb{P}[Z \geq \text{VaR}_\alpha(Z)] \]

2. For $\mathbb{P}[Z > \text{VaR}_\alpha(Z)] \leq \beta < \mathbb{P}[Z \geq \text{VaR}_\alpha(Z)]$ we have that
   \[ \text{VaR}_\alpha(Z) = \text{VaR}_\beta(Z). \]

So, if $Z$ has an atom at $\text{VaR}_\alpha(Z)$, then VaR is blind to some changes at risk level.
Coherent Risk Measures

A risk measure
\[ \rho : \mathcal{M} \to \mathbb{R} \]
\[ Z \to \rho(Z) \]
is said to be **coherent** if it satisfies

- **Translation Equivariance:**
  \[ \rho(Z + a) = \rho(Z) + a, \quad \forall Z, \forall a \in \mathbb{R} \]

- **Monotonicity:**
  \[ \rho(Z) \leq \rho(Y), \quad \text{if } Z \leq Y \]

- **Subadditivity:**
  \[ \rho(Y + Z) \leq \rho(Y) + \rho(Z), \quad \forall Y, Z \]

- **Positive Homogeneity:**
  \[ \rho(aZ) = a\rho(Z), \quad \forall a \geq 0, \forall Z. \]

**Example.** VaR is not coherent.

**Remark**

\[
\begin{bmatrix}
\text{Subadditivity} \\
\text{Positive Homogeneity}
\end{bmatrix} \iff \begin{bmatrix}
\text{Convexity} \\
\text{Positive Homogeneity}
\end{bmatrix}
\]
Two distributions with $VaR_{0.1} = 0$.

$VaR$ is myopic to values worse than itself.
Consider a continuous random variable $Z \in L^1$ and $0 < \alpha < 1$. We define the Average Value at Risk (at level $\alpha$) by

$$AVaR_\alpha(Z) = \mathbb{E}[Z | Z > VaR_\alpha(Z)]$$

$$= \mathbb{E}[Z | Z \geq VaR_\alpha(Z)]$$

$$= \frac{\mathbb{E}[Z \chi_{[Z \geq VaR_\alpha(Z)]}]}{\mathbb{P}[Z \geq VaR_\alpha(Z)]}$$

$$= \frac{1}{\alpha} \mathbb{E}[Z \chi_{[Z \geq VaR_\alpha(Z)]}]$$
AVaR as optimal value

\[ AVaR_\alpha(Z) = VaR_\alpha(Z) + \frac{1}{\alpha} \mathbb{E} \left[ [Z - VaR_\alpha(Z)]^+ \right] \]

Consider the function \( g : \mathbb{R} \to \mathbb{R} \) defined by

\[ g(u) = u + \frac{1}{\alpha} \mathbb{E} \left[ [Z - u]^+ \right] \]

- \( g \) is finite.
- \( g \) is convex.
- \( AVaR_\alpha(Z) = g(VaR_\alpha(Z)) \)
**Proposition**

The function $g$ is differentiable and $g'(VaR_\alpha(Z))$. Thus, $VaR_\alpha(Z)$ is a minimizer of

$$AVaR_\alpha(Z) = \min_u u + \frac{1}{\alpha} \mathbb{E} [(Z - u)^+]$$

For computing AVaR, instead of computing VaR and expected value, you can solve an optimization problem.
Proposition
Given a random variable $Z \in L^1$ and $0 < \alpha < 1$, we have that
1. The function $g(u) = u + \frac{1}{\alpha} \mathbb{E} \left[ (Z - u)^+ \right]$ is finite and convex.
2. The optimization problem
$$\min_u u + \frac{1}{\alpha} \mathbb{E} \left[ (Z - u)^+ \right]$$
always has solution.
3. $\text{VaR}_\alpha(Z)$ is a minimizer of this optimization problem.
For a general random variable $Z \in L^1$ and $0 < \alpha < 1$, we define

$$AVaR_\alpha(Z) := \min_u u + \frac{1}{\alpha} \mathbb{E} [\langle Z - u \rangle^+]$$

- This definition generalizes $AVaR_\alpha(Z) = \mathbb{E}[Z | Z > VaR_\alpha(Z)]$, for continuous random variables.
- The definition as optimal value does not give any clue of its meaning for general distribution. However, it has appealing properties.
- $AVaR_\alpha(Z) \geq VaR_\alpha(Z)$. 
Proposition
AVaR is a coherent risk measure.

Proposition
Given \( Z \in L^1 \).

1. If \( 0 < \alpha \leq \beta < 1 \), then \( AVaR_\beta(Z) \leq AVaR_\alpha(Z) \).
2. \( \lim_{\alpha \uparrow 1} AVaR_\alpha(Z) = \mathbb{E}[Z] = \inf_u u + \mathbb{E}[[Z - u]^+] \).
3. The function \( \alpha \rightarrow AVaR_\alpha(Z) \) is left continuous.

The level \( \alpha \) can be regarded as being a risk tolerance level.