Two-stage Stochastic Programming Problems

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Convex Problems

Consider

$$v(x) = \begin{array}{ll} \inf_y & g(y) \\ \text{s. t.} & y \in \mathcal{G}(x) \end{array}$$

where

- $g(\cdot)$ is convex.
- $\mathcal{G} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is convex (i. e. $t\mathcal{G}(x) + (1-t)\mathcal{G}(z) \subset \mathcal{G}(tx + (1-t)z)$ for all $x, z \in \mathbb{R}^n$ and $t \in [0,1]$)

Example

- Assume $c : \mathbb{R}^m \to \mathbb{R}^n$ is convex. Then $\mathcal{G}(x) = \{y \in \mathbb{R}^m : c(y) \leq x\}$ is convex.
- $\mathcal{G}(z,\xi) = \{y : z + W_\xi y = h_\xi, \quad y \geq 0\}$

Remark. If $\mathcal{G}$ is convex, then $\mathcal{G}(x)$ is also convex.

Convex Problems

Proposition

The multifunction $\mathcal{G}$ is convex if and only if $\text{graph}(\mathcal{G})$ is convex.

Remark: $\mathcal{G}(x) = \text{graph}(\mathcal{G}) \cap (\{x\} \times \mathbb{R}^m)$.

Proposition

If $\mathcal{G}$ and g are convex, then $v : \mathbb{R}^n \to \overline{\mathbb{R}}$ is also convex.

Convex Problems

For the especial case

$$\mathcal{G}(x) = \{y \in \mathbb{R}^m : c(y) \leq x\}$$

where $c : \mathbb{R}^m \to \mathbb{R}^n$ is convex. That is

$$v(x) = \left.\begin{array}{ll} \inf_y & g(y) \\ \text{s. t.} & c(y) \leq x \end{array}\right\} P(x)$$

Remark: If $-\mu \in \partial v(x)$, then $0 \in \partial[v(z) + z^\top \mu](x)$

Convex Problems

The lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n_+ \to \mathbb{R}$ is defined by

$$L(x, y, \mu) = g(y) + (c(y) - x)^\top \mu$$

Definition

For a fixed x, a vector $\bar{\mu} \in \mathbb{R}^n_+$ is a Lagrange multiplier for $\bar{y}$ if

- $\bar{y}$ solves $\min_y L(x, y, \bar{\mu})$
- $0 \leq \bar{\mu} \perp c(\bar{y}) - x \leq 0$

Theorem

1. If $-\bar{\mu} \in \partial v(x)$ and $\bar{y}$ minimizes $P(x)$, then $\bar{\mu}$ is a Lagrange multiplier for $\bar{y}$.
2. If there exists $\bar{\mu}$ Lagrange multiplier for some $\bar{y}$, then $\bar{y}$ is a minimizer for $P(x)$ and $-\bar{\mu} \in \partial v(x)$.

In both cases $v(x) = L(x, \bar{y}, \bar{\mu}) = \inf_y L(x, y, \bar{\mu})$.

Convex Problems

Remark

For $\mu \in \mathbb{R}^n$ we have that

$$\inf_z v(z) + (z - x)^\top \mu = \inf_y L(x, y, \mu) \leq v(x)$$

Definition

The dual function $\theta : \mathbb{R}^n \times \mathbb{R}^b_+ \to \overline{\mathbb{R}}$ by

$$\theta(x, \mu) = \inf_z v(z) + (z - x)^\top \mu = \inf_y L(x, y, \mu) \leq v(x)$$

and the duality gap by

$$v(x) - \sup_{\mu \in \mathbb{R}^n_+} \theta(x, \mu) \geq 0$$

Remark. $\theta(x, \mu) \leq v(x)$ (**Weak Duality**)

Convex Problems

Theorem (Strong Duality)

Assuming that $v(x) \in \mathbb{R}$ and that there exists y such that $c(y) < x$(Slater condition) we have that the dual problem $\max_{\mu \geq 0} \theta(x, \mu)$ has solution $\bar{\mu}$ and $v(x) = \theta(x, \bar{\mu})$ (there is no duality gap).