Two-stage Stochastic Programming Problems
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The Interchangeability Property

Two Stage Problem

First Stage \[
\begin{align*}
\min_x & \quad f(x) + \mathbb{E}[Q(x, \xi)] \\
\text{s. t.} & \quad x \in X
\end{align*}
\]

Second Stage \[
\begin{align*}
Q(x, \xi) := \min_y & \quad g(y, \xi) \\
\text{s. t.} & \quad y \in \mathcal{G}(x, \xi)
\end{align*}
\]

- It is crucial \( Q(x, \cdot) \) to be measurable, for all \( x \).
- Easy case: \( \xi \) discrete random vector (since \( Q(x, \cdot) \) is a discrete random variable).
- Is the second stage minimizer function \( y_{\xi} \) measurable?.
- Example: Consider
  - \( \Omega = [0, 1] \) with the Lebesgue measure.
  - \( \xi(\omega) = \omega \) (uniform distribution)
  - \( \mathcal{G}(x, \xi) = [0, 1] \)
  - \( g(y, \xi) = \text{sign}\left( (\xi - y\chi_N(y))^2 \right) \), where \( N \subset [0, 1] \) is a non measurable set.
The Interchangeability Property

One Stage Problem

\[
\min_{x, y_{\xi}} \quad f(x) + \mathbb{E}[g(y_{\xi}, \xi)] \\
\text{s. t.} \quad x \in X \\
y_{\xi} \in \mathcal{G}(x, \xi), \quad \text{a.e.} \ [\xi] \\
y_{\xi} \text{ is measurable}
\]

\textbf{g}(y_{\xi}, \xi) \text{ needs to be measurable, for all } x.
Proposition

Assuming that $Q(x, \xi)$ and $g(y_{\xi}, \xi)$ are measurable, then the one stage and two stage problems are equivalent.
The Interchangeability Property

**Definition**

A function $g : \mathbb{R}^m \times \Omega \rightarrow \overline{\mathbb{R}}$ is a Carathéodory function if

1. $g(y, \cdot)$ is measurable for all $y$.
2. $g(\cdot, \omega)$ is continuous for a.e. $[\omega]$.

**Theorem**

If $g$ is a Carathéodory function and the probability measure (on $\Omega$), then $g$ is measurable over $\mathbb{R}^m \times \Omega$. 
The Interchangeability Property

Given the problem

\[
G(\omega) := \min_y g(y, \omega) \\
\text{s. t. } c(y, \omega) \leq 0 \quad \text{(P)}
\]

**Theorem**

If all components of \(c(y, \omega)\) are Carathéodory functions, then there exists a family \(\{c^k\}_{k=1}^{\infty}\) of measurable functions such that for each \(\omega\), \(\{c^k(\omega)\}_{k=1}^{\infty}\) is a dense subset of \(\{y : c(y, \omega) \leq 0\}\).

**Theorem**

If \(g(y, \omega)\) and all components of \(c(y, \omega)\) are Carathéodory functions, then \(G(\omega)\) is measurable. Also, if \(E[G] \in \mathbb{R}\), and \(\bar{y}(\omega)\) is a minimizer of (P), then \(\bar{y}\) is measurable and solves

\[
E[G] = \min_y E[g(y, \omega)] \\
\text{s. t. } c(y, \omega) \leq 0 \\
y_\omega \text{ is measurable}
\]