Stochastic Optimal Control Problems
Part I: Deterministic Case

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Outline

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- y state of the system
- u control input

Find a control law and its corresponding trajectory that optimize some performances of the system while complying with prescribed constraints (physical or economical constraints on the control and/or the state)

Consider the problem of minimizing the cost function

$$\int_0^T \ell(y_t, u_t)dt + \phi(y_0, y_T) \quad \text{subject to:} \quad \dot{y}_t = f(y_t, u_t), \; t \in (0, T),$$

and the constraints:

- Control constraints: $c(u_t) \leq 0, \; t \in (0, T)$,
- State constraints: $g(y_t) \leq 0, \; t \in (0, T)$,
- Mixed state and control constraints: $c(u_t, y_t) \leq 0, \; t \in (0, T)$,
- Initial-final equality and inequality constraints:

$$\Phi_i(y_0, y_T) = 0, \quad i = 1, \cdots, r_1,$$
$$\Psi_i(y_0, y_T) \leq 0, \quad i = r_1 + 1, \cdots, r.$$

Function spaces: Control and state spaces

$$\mathcal{U} := L^{\infty}(0, T; \mathbb{R}^m); \quad \mathcal{Y} := W^{1,\infty}(0, T; \mathbb{R}^d).$$

Their extension to Hilbert spaces:

$$\mathcal{U}_2 := L^2(0, T; \mathbb{R}^m); \quad \mathcal{Y}_2 := H^1(0, T; \mathbb{R}^d).$$

The space race: Goddard problem

Example (Goddard)

$$\dot h(t) = v(t), \qquad h(0) = 0,$$
$$\dot v(t) = \frac{u(t)}{m(t)} - g, \qquad v(0) = 0,$$
$$\dot m(t) = -bu(t), \qquad m(0) = m_o$$

$h(t)$: altitude
$v(t)$: velocity
$m(t)$: masse

$u(t)$: thrust

- The trust $u(t)$ is subject to: $0 \le u(t) \le u_{max}$.
- The rocket's mass satisfies the contraint: $m_1 \le m(t) \le m_2(t)$.

The optimal control problem is the following:

$$\left\| \begin{array}{l} \text{Max } h(T) \\ u(t) \in [0, u_{\max}], \quad (h, v, m) \text{ vérifie l'EDO,} \\ m_1 \le m(t) \le m_2(t) \quad t \ge 0. \end{array} \right.$$

Launcher's problem: Ariane 5

- Steer the launcher from Kourou to the GEO
- State variables $(\mathbf{r}, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \overrightarrow{P} + \overrightarrow{F_T}(\mathbf{r}, \mathbf{v}, u) - \overrightarrow{F_D}(\mathbf{r}, \mathbf{v}, u); \end{aligned}$$

$u \in \mathbb{R}^3$ the trust force (control input).

- State constraints: Heat flux, limited capacity of ergol, target constraint (GEO)

Objective function: maximization of the payload.

Standing assumptions

Assume the set of admissible control inputs is:

$$\mathcal{U}_{\rm ad} := \{u \in \mathcal{U}; u_t \in U \text{ on } (0,T)\}.$$

(A0) U is a closed set in $\mathbb{R}^m$.

(A1) $f : \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^d$ is loc. Lipschitz continuous.

(A2) For every $x \in \mathbb{R}^d$, $f(x,U)$ is a convex set of $\mathbb{R}^d$.

Proposition

Assume **(A0)-(A1)**. Let $x \in \mathbb{R}^d$.

i) For every $u \in \mathcal{U}_{\rm ad}$, there exists $y^u \in H^1([0,T];\mathbb{R}^d)$ solution of the equation: $\dot{y}_t^u = f(y_t^y, u_t), \qquad y_0^u = x.$

ii) Moreover, the application defined by

$$\begin{array}{rcl} \mathcal{T}(\cdot) : L^2(0,T;\mathbb{R}^m) & \longrightarrow & H^1(0,T;\mathbb{R}^d) \\ u & \longmapsto & \mathcal{T}(u) := y^u \end{array}$$

is continuous

$$\mathcal{S}_{[0,T]}(x) := \left\{ y \mid \exists u \in \mathcal{U}_{\rm ad}, \;\; \dot{y}_t = f(y_t, u_t), \qquad y_0 = x \right\}$$

Under **(A0)-(A2)** and if U is a compact set,

- $\mathcal{S}_{[0,T]}(x)$ is a compact set in $W^{1,1}$ endowed with C^0-topology.
  This result is a consequence of Filippov's theorem, see the books of Vinter (2010) or Aubin-Cellina (1984).

- the set-valued function $x \rightsquigarrow \mathcal{S}_{[0,T]}(x)$ is Lipschitz continuous,

$$\exists L > 0, \mathcal{S}_{[0,T]}(x) \subset \mathcal{S}_{[0,T]}(z) + L|x - z|\mathcal{B}_{W^{1,1}} \quad \forall x, z \in \mathbb{R}^d.$$

Example (1)

$$\text{Min} \int_0^1 y^2(t)\,dt$$
$$\dot{y}(t) = u(t),$$
$$y(0) = 0,$$
$$u(t) \in \{-1, 1\}$$

- $u_n(t) = \begin{cases} 1 & \text{sur } \left(\frac{2k}{2n}, \frac{2k+1}{2n}\right) \\ -1 & \text{sur } \left(\frac{2k+1}{2n}, \frac{2k+2}{2n}\right) \end{cases}$
- $y_n(t) = \begin{cases} t - \frac{k}{n} & \text{sur } \left(\frac{2k}{2n}, \frac{2k+1}{2n}\right) \\ -t + \frac{(k+1)}{n} & \text{sur } \left(\frac{2k+1}{2n}, \frac{2k+2}{2n}\right) \end{cases}$

This simple problem doesn't admit a solution

$y_n \to 0, \qquad y \equiv 0$ is not admissible !!

$\|u_n\|_{L^\infty, L^2} = 1 \not\to 0$

Example (1')

$$\text{Min} \int_0^1 y^2(t)\,dt$$
$$\dot{y}(t) = u(t),$$
$$y(0) = 0,$$
$$u(t) \in [-1, 1]$$

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The relaxed control problem admits a solution!

$y_n \to 0, \qquad y \equiv 0$ is admissible

$\|u_n\|_{L^\infty, L^2} = 1 \not\to 0$

Outline

"First discretize and then optimize"

Consider a general control problem

$$
\begin{aligned}
&\text{Min } \phi(y_T) + \int_0^T \ell(y_t, u_t) \\
&\text{subject to:} \quad \dot{y}_t = f(y_t, u_t),\ t \in (0, T), \quad y_0 = x \\
&\qquad\qquad\qquad c(u_t) \le 0,\ t \in (0, T), \\
&\qquad\qquad\qquad g(y_t) \le 0,\ t \in (0, T), \\
&\qquad\qquad\qquad c(u_t, y_t) \le 0,\ t \in (0, T), \\
&\qquad\qquad\qquad \Phi_i(y_0, y_T) = 0, \quad i = 1, \cdots, r_1, \\
&\qquad\qquad\qquad \Psi_i(y_0, y_T) \le 0, \quad i = r_1 + 1, \cdots, r.
\end{aligned}
$$

The Euler discretization

- N: number of time steps, $h_k > 0$ duration of k-th time step
- Steps begin at time $t_0 = 0$, and for $k = 1$ to N, $t_k = \sum_{j=0}^{k} h_j$
- State equation: $y_{k+1} = y_k + h_k f(u_k, y_k), \;\; k = 0, \cdots, N-1.$
- Cost function: $\phi(y_N)+$
- Running constraints:

$$c(u_k) \leq 0; g(y_k) \leq 0; c(u_k, y_k) \leq 0, \quad k = 1, \cdots, N-1.$$

- Final equality and inequality constraints:

$$\Phi_i(y_0, y_N) = 0, \quad i = 1, \cdots, r_1,$$
$$\Psi_i(y_0, y_N) \leq 0, \quad i = r1 + 1, \cdots, r.$$

- Some control problems are "naturally" desribed by controlled discrete dynamics.
- Indeed, in some cases the control can act on the control variable only at very specific dates (daily, monthly, ...)
- In this case, the time schedule is fixed and the control problem is already in the form of a complex finite dimensional control problem.

Example: A production problem

- y_t: amount of steel produced at time t.
- $0 \leq u_t \leq 1$ is a fraction of steel produced at time t and allocated to investment.
- Thepart of y_t allocated to investment is used to increase the production capacity according to Eq:

$$\frac{dy_t}{dt}(t) = k u_t y_t,$$

where $y_0 = A$ is the initial production and k is the coefficient of increase in production.

The optimal control problem consists here at choosing u in an optimal way to **maximize** the production allocate to the consumption during a fixed time horizon T.

Questions

In case of continuous control problem

How is the discretized version related to the original continuous control problem ?

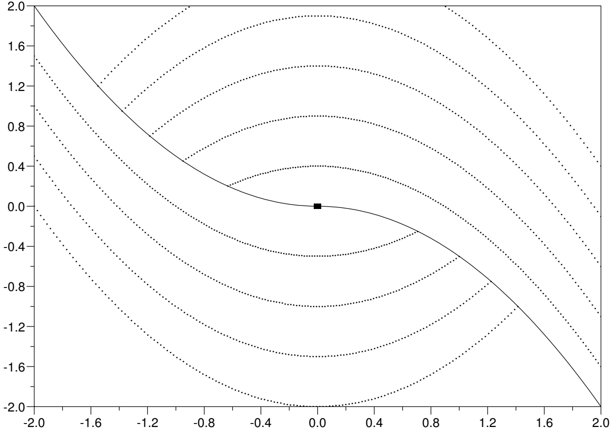
Given a nominal local solution $(\bar{u}, \bar{y})$ of the original problem:

- Does the discretized problem have a solution (u_h, y_h) near $(\bar{u}, \bar{y})$??
- Can we expect an Error order as $\|u_h - \bar{u}\| + \|y_h - \bar{y}\| = O(h)$, where $h := \max_k h_k$?
- Is it reasonable to assume that the solution is (piecewise) smooth ?
- How do we solve the discretized problem ?

Example: double integrator (I)

Consider the very simple example with constraints on the control:

- Dynamics: $\ddot{y}_t = u_t \in [-1, 1]$
- Optimization problem: reach the zero state in minimal time



Example: double integrator (I)

- Solution: Bang-bang optimal control, at most one switching time
- Discretized solution of same nature (costate affine function of time)
- Error only due to the switching time step
- Expected error: at most $O(h)$

Ref. Alt, Baier, Gerdts, Lempio, *Error bounds for Euler approximation of linear-quadratic control problems with bang-bang solutions*. 2012.

Example: double integrator (II)

Same dynamics: $\ddot{x}_t = u_t \in [-1, 1]$; Integral cost $\int_0^T x_t^2 \mathrm{d}t$.

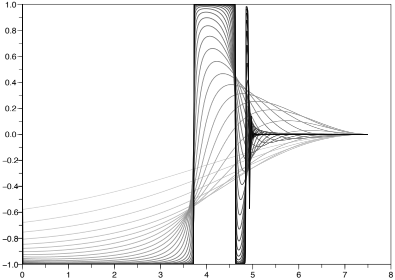


Figure 2: Fuller problem: optimal control, logarithmic penalty

Ref. PhD work of J. Laurent-Varin, 2005.

PROS

- This method can integrate all types of constraints (state constraints, mixed constraints, ... etc)
- The discrete problem is a finite dimensional optimisation problem

CONS

- local approach
- Huge number of variables
- Stability and convergence results: in some cases, the discretized control problem doesn't have any feasible solution while the original control problem does have a solution!
- The discretization of the control problem should take into account the structure of the optimal trajectory

Outline

With a final state constraint.

$$\text{Min } \phi(y_T)$$

$$\text{subject to:} \quad \dot{y}_t = f(y_t, u_t), \; t \in (0, T), \quad y_0 = y_0$$

$$\Psi(y_T) = 0$$

The mapping $\mathcal{T} : u \longmapsto y_u$ is univoque

The OCP $(\mathcal{P})$ can be re-written as:

$$\text{Min} \mathcal{F}(u) := \mathcal{J}(u, y_u)$$
$$u \in \mathcal{U}_{ad}; \; \Psi(\mathcal{T}(u))(T) = 0.$$

Reminder (A known result in Optimization theory)

$\bar{u} \in \mathcal{U}_{ad}$ *is a minimum of* $(\mathcal{P}) \implies$
$\exists(\lambda_o, \lambda) \neq 0, \; [\lambda_o \mathcal{F}'(\bar{u}) + [\Psi'(\mathcal{T}(u)) \cdot \mathcal{T}'(\bar{u})(T)]^\mathsf{T} \lambda] \cdot (u - \bar{u}) \geq 0$
$\forall u \in \mathcal{U}_{ad}$.

Differentiability of $\mathcal{F}$

(A1') Assume f is of classe C^1.

Theorem

Assume (A0)-(A1) and (A1'), then $\mathcal{T}$ is differentiable on $L^2(0,T;\mathbb{R}^m)$. Moreover, we have :

$$\mathcal{T}'(u)\cdot v = z_v^u \qquad \forall u,v\in L^2(0,T;\mathbb{R}^m);$$

where z_v^u is the linearized state, solution of:

$$\begin{cases} \dot z_t = f_y'(y_t^u,u_t)z_t + f_u'(y_t^u,u_t)v_t & \text{on } (0,T),\\ z_0 = 0, \end{cases} \tag{1}$$

where $y_\cdot^u := \mathcal{T}(u)$ stands for the state associated to u.

Theorem

We have:

$$\lambda_o \mathcal{F}'(u).v + [\mathcal{T}'(\bar{u})(T)]^{\mathsf{T}}\lambda] \cdot v = \int_0^T \langle p(t), f_u(y_t^u, u_t) \cdot v_t)\rangle \, dt$$

where $y^u = \mathcal{T}(u)$*, and* p *is the adjoint state associated to* u*, solution of:*

$$-\dot{p}(t) = [f_y(y_t^u, u_t]^t p(t),$$
$$p(T) = \lambda_o \Phi'(T, y_T^u) + \lambda$$

Itroduce the hamiltonien $H : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$, defined by:

$$H(x, q, v) = q \cdot f(x, v).$$

Theorem (Sous (A1)-(A3) et (A1'))

let $\bar{u} \in \mathcal{U}_{ad}$ is a minimum of $(\mathcal{P})$, then the triplet $(\bar{u}, \bar{y}, \bar{p})$ satifies:

$$\begin{aligned} &\dot{\bar{y}}(t) = f(\bar{y}(t), \bar{u}(t)), \quad \bar{y}(0) = x_o \\ &-\dot{\bar{p}}(t) = [f_y(\bar{y}_u(t), \bar{u}(t))]^t p(t), \\ &\partial_u \mathcal{H}(\bar{y}(t), \bar{u}(t), \bar{p}(t)) \cdot (u - \bar{u}(t)) \geq 0, \quad \forall u \in U. \end{aligned}$$

.

The triplet $(\bar{u}, \bar{y}, \bar{p})$ is called a Pontryagin extremal.

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$$-\dot{\bar{p}}(t) = [f_y(\bar{y}_u(t), \bar{u}(t))]^t p(t),$$

$$\mathcal{H}(\bar{y}(t), \bar{u}(t), \bar{p}(t)) = \min_{u \in U} \mathcal{H}(\bar{y}(t), u, \bar{p}(t)).$$

The triplet $(\bar{u}, \bar{y}, \bar{p})$ is called a Pontryagin extremal.

More generally ...

$$\text{Min } \phi(y_T) + \int_0^T \ell(y(t), u(t))\, dt$$

$$\text{subject to:} \quad \dot{y}_t = f(y_t, u_t),\ t \in (0, T), \quad y_0 = y_0$$

$$\Psi(y_T) = 0$$

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let $\bar{u} \in \mathcal{U}_{ad}$ is a minimum of $(\mathcal{P})$, then there exists $(\lambda_0, \lambda) \in \{0,1\} \times \mathbb{R}^d$ such that

$$\dot{\bar{y}}(t) = \partial_p H(\bar{y}(t), \bar{u}(t), \bar{p}(t), \lambda_0), \quad \bar{y}(0) = x_o$$
$$-\dot{\bar{p}}(t) = \partial_y H(\bar{y}_u(t), \bar{u}(t), \bar{p}(t), \lambda_0)]^t p(t),$$
$$\partial_u \mathcal{H}(\bar{y}(t), \bar{u}(t), \bar{p}(t), \lambda_0) \cdot (u - \bar{u}(t)) \geq 0, \quad \forall u \in U,$$

where $H(x, v, q, \mu) := \langle q, f(x, a) \rangle + \mu \ell(x, v)$ for $x \in \mathbb{R}^d$, $v \in U$, $q \in \mathbb{R}^d$ and $\mu \in \{0, 1\}$.

Moreover, $\lambda_o = 1$ if the problem is free of state constraints.

More generally ...

$$\text{Min } \phi(y_T) + \int_0^T \ell(y(t), u(t))\, dt$$

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\end{aligned}$$

where $H(x, v, q, \mu) := \langle q, f(x, a) \rangle + \mu \ell(x, v)$ for $x \in \mathbb{R}^d$, $v \in U$, $q \in \mathbb{R}^d$ and $\mu \in \{0, 1\}$.

Moreover, $\lambda_o = 1$ if the problem is free of state constraints.