—— TUTORIAL ——
RISK, OPTIMIZATION AND STATISTICS

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Optimization and Statistical Estimation

support for decision-making in a stochastic environment

Optimization:

minimize a "cost" expression under constraints on the decision

the constraints could involve bounds on other "costs"

the "costs" may have a background in statistical analysis

Statistical Estimation:

approximate some quantity from empirical/historical data

minimize an error expression to get regression coefficients

different interpretations of "error" yield different results

Interplay:

- optimization problems involving uncertainty depend on estimation methodology even in coming to a formulation
- estimation problems are optimization of a special sort
- new and deeper connections are now coming to light

The Risk Quadrangle — A New Paradigm

an array of "quantifications" be applied to random variables X
"cost" orientation of X: high outcomes bad, low outcomes good

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$$\begin{array}{ccc} & \text{risk } \mathcal{R} \longleftrightarrow \mathcal{D} \text{ deviation} & \\ \text{optimization} & \uparrow\downarrow \quad \mathcal{S} \quad \downarrow\uparrow & \text{statistics} \\ & \text{regret } \mathcal{V} \longleftrightarrow \mathcal{E} \text{ error} & \end{array}$$

$\mathcal{R}(X)$ elicits the "level of cost" in X to use in making comparisons
$\mathcal{V}(X)$ quantifies the "anti-utility" in outcomes $X > 0$ versus $X \leq 0$
$\mathcal{D}(X)$ measures the "nonconstancy" in X as its uncertainty
$\mathcal{E}(X)$ measures the "nonzeroness" in X for use in estimates
$\mathcal{S}(X)$ is a "statistic" associated with X through $\mathcal{E}$ and through $\mathcal{V}$

Background Source

$\longrightarrow$ for references, details and examples behind this tutorial

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R.T. Rockafellar and S.P. Uryasev (2013),
"The fundamental risk quadrangle in risk management,
optimization and statistical estimation,"
Surveys in Management Science and O.R. 18, 33–53.

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downloads: www.math.washington.edu/~rtr/mypage.html
(look for item #218)

Uncertain "Costs" / "Losses" / "Damages"

"Costs": quantities to be minimized or kept below given levels
(this fits better with optimization conventions than "profits")

General "cost" expression in decision-making:
$c(x, v)$ with $x =$ **decision** vector, $v =$ **data** vector

$$x = (x_1, \ldots, x_n), \qquad v = (v_1, \ldots, v_m)$$

Stochastic uncertainty:
v is replaced by a **random variable** vector $V = (V_1, \ldots, V_m)$
then "cost" becomes a **random variable**: $\underline{c}(x) = c(x, V)$

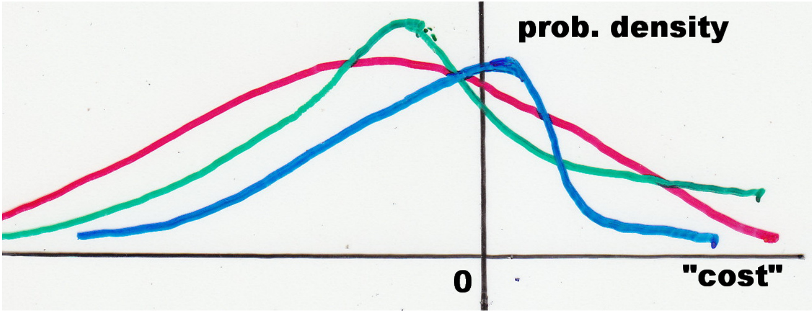
Portfolio example in finance:
$V_j =$ random return on asset j, $x_j =$ amount of j in portfolio
$\underline{c}(x) = c(x, V) = -[x_1 V_1 + \cdots + x_n V_n]$ (random loss incurred)

Design examples in engineering: "cost" $\longleftrightarrow$ "hazard"

Challenges in Optimization Modeling

"cost" $\underline{c}(x)$ = random variable depending on decision x

outcomes < 0, if any, correspond to "rewards"



Key issue in problem formulation

- the distribution of $\underline{c}(x)$ can only be shaped by the choice of x
- but how then can constraints and minimization be understood?

A Broad Pattern for Handling Risk in Optimization

Risk measures: functionals $\mathcal{R}$ that "**quantify the risk**" in a random variable X by a numerical value $\mathcal{R}(X)$

("risk" $\neq$ "uncertainty")

Systematic prescription

Faced with an uncertain "cost" $\underline{c}(x) = c(x, V)$ articulate it numerically as $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$ for a choice of risk measure $\mathcal{R}$

Constraints: keeping $\underline{c}(x)$ "acceptably" $\leq b$

modeled as: constraint $\bar{c}(x) = \mathcal{R}(\underline{c}(x)) \leq b$

Objectives: making $\underline{c}(x)$ as "acceptably" low as possible

modeled as: minimizing $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$

i.e., minimizing the threshold level b such that x can be selected with $\underline{c}(x)$ "acceptably" $\leq b$

Stochastic Framework

Space of future states: Ω with elements ω ("scenarios")

$\mathcal{A}_0$ = field of subsets of Ω, P_0 = probability measure on $\mathcal{A}_0$

Random variables: functions $X : \Omega \to R$ ($\mathcal{A}_0$-measurable)

$$EX = \mu(X) = \int_\Omega X(\omega) dP_0(\omega), \quad \sigma^2(X) = E[(X - EX)^2]$$

Function space setting: $X \in \mathcal{L}^2 := \mathcal{L}^2(\Omega, \mathcal{A}_0, P_0)$ for simplicity

Hilbert space of random variables with finite mean and variance

$$\langle X, Y \rangle = E[XY], \qquad ||X|| = \sqrt{E[X^2]}$$

Treatment of probability alternatives

Measures P can be represented by densities $\frac{dP}{dP_0}$ with respect to P_0

$$E_P(X) = \int_\Omega X(\omega) \frac{dP}{dP_0}(\omega) dP_0(\omega) = \langle X, Q \rangle \text{ for } Q = \frac{dP}{dP_0}$$

[sets $\mathcal{P}$ of alternatives P] $\longleftrightarrow$ [sets $\mathcal{Q}$ of $Q \in \mathcal{L}^2$: $Q \geq 0$, $EQ = 1$]

Axiomatization of Risk

Axioms for <u>regular</u> measures of risk: $\mathcal{R} : \mathcal{L}^2 \to (-\infty, \infty]$

(R1) $\mathcal{R}(C) = C$ for all constants C

(R2) convexity, (R3) closedness (lower semicontinuity)

(R4) aversity: $\mathcal{R}(X) > EX$ for nonconstant X

Additional properties of major interest:

(R5) positive homogeneity: $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ when $\lambda > 0$

(R6) monotonicity: $\mathcal{R}(X) \leq \mathcal{R}(X')$ when $X \leq X'$

Coherent measures of risk: $\mathcal{R}$ satisfying (R1), (R2), (R5), (R6)

Artzner et al. (2000) introduced coherency without aversity

Note: (R1)+(R2) $\implies \mathcal{R}(X + C) = \mathcal{R}(X) + C$ for constants C

Preservation of convexity under (R1)+(R2)+(R6)

$$\underline{c}(x) = c(x, V) \text{ convex in } x \quad \implies \quad \bar{c}(x) = \mathcal{R}(\underline{c}(x)) \text{ convex in } x$$

Some Common Approaches From This Perspective

Best guess of future state: $\mathcal{R}(X) = X(\bar{\omega})$ $(\mathrm{prob}(\bar{\omega}) > 0)$

then $\mathcal{R}(X) \leq b \iff X(\bar{\omega}) \leq b$

$\longrightarrow$ coherent but not averse (and lacking any ability to hedge)

Focusing on worst cases: $\mathcal{R}(X) = \sup X$ (ess. sup)

then $\mathcal{R}(X) \leq b \iff X \leq b$ almost surely

$\longrightarrow$ coherent, averse (but perhaps overly conservative, infeasible)

Passing to expectations: $\mathcal{R}(X) = \mu(X) = EX$

then $\mathcal{R}(X) \leq b \iff X \leq b$ "on average"

$\longrightarrow$ coherent but not averse (perhaps too feeble)

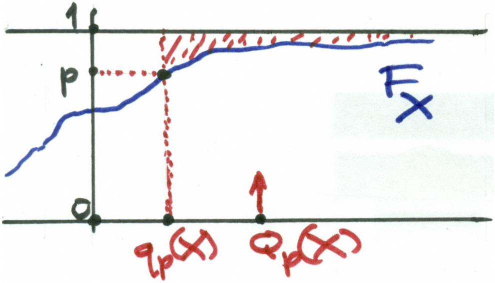
Adopting a safety margin: $\mathcal{R}(X) = \mu(X) + \lambda\sigma(X)$ $\lambda > 0$

then $\mathcal{R}(X) \leq b$ unless in tail $> \lambda$ standard deviations

$\longrightarrow$ averse but not coherent (lacks monotonicity!)

Quantiles and "Superquantiles": VaR and CVaR

F_X = cumulative distribution function for random variable X



p-**Quantile:** "value-at-risk" in finance

$$q_p(X) = \mathrm{VaR}_p(X) = \text{“}F_X^{-1}(p)\text{”}$$

p-**Superquantile:** "conditional value-at-risk" in finance

$$Q_p(X) = \mathrm{CVaR}_p(X) = \text{“}E[\,X\,|X \geq q_p(X)\,]\text{”} = \frac{1}{1-p}\int_p^1 q_t(X)dt$$

mathematical behavior: quantiles bad, superquantiles good

Measures of Risk Based on Probability Thresholds

Looking at quantiles/VaR: $\mathcal{R}(X) = q_p(X)$

then $\mathcal{R}(X) \leq b \iff \text{prob}\{X \leq b\} \geq p$

$\longrightarrow$ *not* coherent, *not* averse (troublesome, subject to criticism)

Superquantiles/CVaR instead: $\mathcal{R}(X) = Q_p(X)$

then $\mathcal{R}(X) \leq b \iff \underline{c}(x) \leq b$ on average in upper p-tail

$\longrightarrow$ coherent, averse (easy to work with and more conservative!)

Corresponding concepts of "failure"

$\underline{c}(x) = c(x, V)$ captures "hazard," failure $\longleftrightarrow$ outcomes > 0

$q_p(\underline{c}(x)) \leq 0$ means **ordinary** probability of failure is $\leq 1 - p$

$Q_p(\underline{c}(x)) \leq 0$ means **buffered** probability of failure is $\leq 1 - p$

Example: case of $p = 0.9$, focusing on the worst 10% of events

$\longrightarrow$ buffered probability of failure ≤ 0.1 means: even in that tail range, the hazard variable comes out "safe on average"

Minimization Formula for VaR and CVaR

$$\mathrm{CVaR}_p(X) = \min_{C \in R} \left\{ C + \frac{1}{1-p} E[\max\{0, X - C\}] \right\} \text{ for } p \in (0,1)$$
$$\mathrm{VaR}_p(X) = \mathrm{argmin} \quad \text{(if unique, otherwise the lowest)}$$

Application to CVaR models: convert a problem in x like

minimize $\mathrm{CVaR}_{p_0}(\underline{c}_0(x))$ subject to [basic constraints and]
$\mathrm{CVaR}_{p_i}(\underline{c}_i(x)) \leq b_i$ for $i = 1, \ldots, m$

into a problem in x and auxiliary variables $C_0, C_1, \ldots, C_m$,

minimize $C_0 + \frac{1}{1-p_0} E[\max\{0, \underline{c}_0(x) - C_0\}]$ while requiring
$C_i + \frac{1}{1-p_i} E[\max\{0, \underline{c}_i(x) - C_i\}] \leq b_i, \quad i = 1, \ldots, m$

Important case: this converts to **linear programming** when
(1) each $\underline{c}_i(x) = c_i(x, V)$ depends **linearly** on x,
(2) the future state space Ω is modeled as **finite**

Some CVaR/Superquantile References

[1] R.T. Rockafellar and S.P. Uryasev (2000),
" Optimization of Conditional Value-at-Risk,"
Journal of Risk 2, 21–42.

[2] R.T. Rockafellar and S.P. Uryasev (2002),
" Conditional Value-at-Risk for General Loss Distributions,"
Journal of Banking and Finance 26, 1443–1471.

[3] R. T. Rockafellar, J. O. Royset (2010),
"On Buffered Failure Prob. in Design and Optim. of Structures,"
Journal of Reliability Engineering and System Safety 95, 499–510.

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downloads: www.math.washington.edu/∼rtr/mypage.html
look for items #179, #187, #211

Stochastic Ambiguity, Admitting Alternative Probabilities

Probability density functions: $Q \in \mathcal{L}^2$ with $Q \geq 0$, $EQ = 1$

$\implies Q = \frac{dP}{dP_0}$ for some probability measure P

$$E_P(X) = \langle X, Q \rangle = E[XQ] = \int_\Omega X(\omega) \frac{dP}{dP_0}(\omega) dP_0(\omega)$$

the underlying probability measure P_0 corresponds to $Q \equiv 1$

Stochastic ambiguity: not trusting just P_0, looking at other P

interested in $\sup_{P \in \mathcal{P}} E_P(X)$ instead of just $EX = E_{P_0}(X)$

Risk envelopes: sets $\mathcal{Q} \subset \mathcal{L}^2$ consisting of probability densities Q

interested in $\mathcal{R}(X) = \sup_{Q \in \mathcal{Q}} E[XQ]$ as a **measure of risk**

note: $\mathcal{R}$ is unaffected if $\mathcal{Q}$ replaced by its closed convex hull

Regularity of a risk envelope:

$\mathcal{Q}$ is closed convex $\neq \emptyset$ and $1 \in \mathcal{Q}$, but 1 isn't a "support point"

(i.e., $\nexists$ hyperplane touching $\mathcal{Q}$ at $Q \equiv 1$ without $\supset \mathcal{Q}$)

Dualization of Monotonic Measures of Risk

Risk envelope characterization, positively homogeneous case

$$\mathcal{R}(X) = \sup_{Q\in\mathcal{Q}} E[XQ], \qquad \mathcal{Q} = \{Q \mid E[XQ] \le \mathcal{R}(X),\ \forall X\}$$

furnishes a **one-to-one correspondence** between

(a) regular risk measures $\mathcal{R}$ that are monotonic + pos. homog.

(b) regular risk envelopes $\mathcal{Q}$, as above

Risk envelope characterization, general case

$$\mathcal{R}(X) = \sup_{Q\in\mathcal{Q}} \{E[XQ] - \mathcal{J}(Q)\},$$
$$\mathcal{J}(Q) = \sup_{X\in\mathcal{L}^2} \{E[XQ] - \mathcal{R}(Q)\}, \qquad \mathcal{Q} = \mathrm{cl}(\mathrm{dom}\,\mathcal{J})$$

furnishes a **one-to-one correspondence** between

(a) regular risk measures $\mathcal{R}$ that are monotonic

(b) regular risk envelopes $\mathcal{Q}$ as $\mathrm{cl}(\mathrm{dom}\,\mathcal{J})$ for a closed convex functional $\mathcal{J} : \mathcal{L}^2 \to [0,\infty]$ such that $\mathcal{J}(1) = 0 = \min \mathcal{J}$

$\mathcal{J}(Q)$ assesses the "**divergence**" of $Q = \frac{dP}{dP_0}$ from $1 = \frac{dP_0}{dP_0}$

Some Examples of Risk Dualization

Risk measures with positive homogeneity:

- for $\mathcal{R}(X) = Q_p(X) = \mathrm{CVaR}_p(X)$ the risk envelope is:
  $\mathcal{Q} = \{Q \,|\, Q \geq 0,\ EQ = 1,\ Q \leq \frac{1}{1-p}\}$
- for $\mathcal{R}(X) = \sup X$ the risk envelope is:
  $\mathcal{Q} = \{Q \,|\, Q \geq 0,\ EQ = 1\}$ (the full "**probability simplex**")

Risk measures without positive homogeneity:

$\mathcal{R}(X) = \log E[\exp X]$ is a regular measure of risk that is also monotonic but **not** positively homogeneous. Its risk envelope is $\mathcal{Q}$ = probability simplex supplied with $\mathcal{J}(Q) = -E[Q \log Q]$

Note: for $Q = dP/P_0$, the Bolzano-Shannon entropy expression

$$-E[Q \log Q] = -\int_\Omega \left[\frac{dP}{dP_0}(\omega) \log \frac{dP}{dP_0}(\omega)\right] dP_0(\omega)$$

is known as the Kullback-Leibler divergence of P from P_0.

"Robust" Optimization Revisited and Refined

Motivation behind so-called "robust" optimization:

- probabilities are often hard to assess, even as guesswork
- they can be avoided by focusing on $\mathcal{R}(X) = \sup_{\omega \in \Omega} X(\omega)$

Practical compromise/pitfall: this depends on the model for Ω

- which "scenarios" ω should go into Ω as the uncertainty set?
- subjective probability enters in deciding which ones to exclude

Nested robustness

Let Ω be **partitioned** into $\Omega_1, \ldots, \Omega_N$ with $p_k = \mathrm{prob}[\Omega_k]$, and let

$$\mathcal{R}(X) = \sum_{k=1}^{N} p_k \sup_{\omega \in \Omega_k} X(\omega).$$

Then $\mathcal{R}$ is regular, monotonic, pos. homogeneous, with envelope

$$\mathcal{Q} = \left\{ Q \,\middle|\, Q \geq 0,\ \mathrm{prob}_Q[\Omega_k] = p_k,\ \forall\, k \right\},$$

Alternative interpretation:

- the partition generates an information field $\mathcal{A}$
- $\mathcal{Q} \longleftrightarrow$ all prob. measures P consistent with that information

Recalling the Risk Quadrangle for the Next Development

$$\begin{array}{ccc} & \text{risk } \mathcal{R} \longleftrightarrow \mathcal{D} \text{ deviation} & \\ \text{optimization} & \uparrow\downarrow \quad \mathcal{S} \quad \downarrow\uparrow & \text{statistics} \\ & \text{regret } \mathcal{V} \longleftrightarrow \mathcal{E} \text{ error} & \end{array}$$

$\mathcal{R}(X)$ elicits the "level of cost" in X to use in making comparisons
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$\mathcal{S}(X)$ is a "statistic" associated with X through $\mathcal{E}$ and through $\mathcal{V}$

Regret Versus Utility

Regret: the "compensation" $\mathcal{V}(X)$ for facing a future cost/loss X
in contrast to the "utility" $\mathcal{U}(Y)$ perceived in a future gain Y

$$\mathcal{V}(X) = -\mathcal{U}(-X) \quad \longleftrightarrow \quad \mathcal{U}(Y) = -\mathcal{V}(-Y)$$

Axioms for regular measures of regret: $\mathcal{V} : \mathcal{L}^2 \to (-\infty, \infty]$

(V1) $\mathcal{V}(0) = 0$, (V2) convexity, (V3) closedness

(V4) aversity: $\mathcal{V}(X) > EX$ for nonconstant X

Additional properties of major interest:

(V5) positive homogeneity: $\mathcal{V}(\lambda X) = \lambda \mathcal{V}(X)$ when $\lambda > 0$

(V6) monotonicity: $\mathcal{V}(X) \leq \mathcal{V}(X')$ when $X \leq X'$

$\Longrightarrow$ axioms for **regular measures of utility** $\mathcal{U} : \mathcal{L}^2 \to [-\infty, \infty)$

(more explanation of utility connections will come later)

regret is oriented to minimizing, utility is oriented to maximizing

Risk From Regret

Goal: generalize to other risk measures the superquantile formula

$$Q_p(X) = \min_{C \in R}\left\{C + \frac{1}{1-p}E[\max\{0, X - C\}]\right\}, \quad q_p(X) = \text{argmin}$$

Trade-off Theorem

For any regular measure of regret $\mathcal{V}$, the formula

$$\mathcal{R}(X) = \min_{C \in R}\{C + \mathcal{V}(X - C)\}$$

defines a regular measure of risk $\mathcal{R}$ such that

$$\mathcal{V} \text{ monotonic} \implies \mathcal{R} \text{ monotonic}$$

$$\mathcal{V} \text{ pos. homog.} \implies \mathcal{R} \text{ pos. homog.}$$

Trade-off interpretation:

C = "designated loss" (to be written off here and now)

$X - C$ = "residual loss" (still to be faced in the future)

Application to insurance: the argmin leads to the "premium"

Optimization role: simplifying $\mathcal{R}$ to $\mathcal{V}$ in objective/constraints

More About Utility

Finance question: for random variables Y representing monetary gains, how to think of one as being preferable to another?

Traditional approach through expected utility

- there is a utility function u to apply to money amounts y
- the functional $\mathcal{U} : Y \rightarrow \mathcal{U}(Y) = E[u(Y)]$ is then the key:

Y_1 is preferred (strictly) to Y_2 $\iff$ $\mathcal{U}(Y_1) > \mathcal{U}(Y_2)$

Background: von Neumann/Morgenstern theory for "lotteries"

the utility function u is **concave** and **nondecreasing**,
and can be **normalized** to have $u(0) = 0$ and $u(y) \leq y$

$\Longrightarrow$ a benchmark focus with utility scaled to money, in which
$\mathcal{U}$ is concave, nondecreasing, with $\mathcal{U}(0) = 0$ and $\mathcal{U}(Y) \leq E[Y]$

Regular measures of utility: such $\mathcal{U}(Y)$ more generally

Utility and Regret in the Monotonic Expectational Case

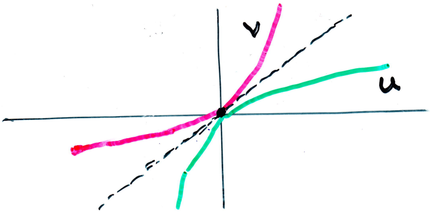
Expected utility: $\mathcal{U}(Y) = E[u(Y)]$ (*normalized* u)

$u(y)$ concave, nondecreasing with $u(0) = 0$, $u(y) \leq y$

Expected regret: $\mathcal{V}(X) = E[v(X)]$

$v(x)$ convex, nondecreasing with $v(0) = 0$, $v(x) \geq x$

$$v(x) = -u(-x) \quad \longleftrightarrow \quad u(y) = -v(-y)$$



Superquantile formula example: $\mathcal{V}(X) = \frac{1}{1-p} E[\max\{0, X\}]$

$$v(x) = \frac{1}{1-p} \max\{0, x\}, \qquad u(y) = \frac{1}{1-p} \min\{0, y\}$$

Measures of Utility Beyond Simple Expected Utility

Utility reflecting stochastic ambiguity

- Let u be a nondecreasing concave utility function, normalized
- For a risk envelope $\mathcal{Q}_0$ and an associated divergence $\mathcal{J}_0$, let

$$\mathcal{U}(Y) = \inf_{Q \in \mathcal{Q}_0} \{ E[u(Y)Q] + \mathcal{J}_0(Q) \}$$

Then $\mathcal{U}$ is a regular measure of utility that is monotonic:

$\mathcal{U}$ is concave, nondecreasing, with $\mathcal{U}(0) = 0$ and $\mathcal{U}(Y) \leq E[Y]$

Corresponding ambiguity version of regret

- Let v be a nondecreasing convex regret function
- For a risk envelope $\mathcal{Q}_0$ and an associated divergence $\mathcal{J}_0$, let

$$\mathcal{V}(X) = \sup_{Q \in \mathcal{Q}_0} \{ E[v(X)Q] - \mathcal{J}_0(Q) \}$$

Then $\mathcal{V}$ is a regular measure of regret that is monotonic

Relation to risk: for the risk measure $\mathcal{R}_0$ dual to $\mathcal{Q}_0$, $\mathcal{J}_0$,

$$\mathcal{U}(Y) = -\mathcal{R}_0(-u(Y)), \quad \mathcal{V}(X) = \mathcal{R}_0(v(X))$$

Passing Now to the Statistics Side of the Risk Quadrangle

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Deviation as a Quantification of Uncertainty

$\mathcal{D}(X)$ generalizes standard deviation $\sigma(X)$

Axioms for regular measures of deviation: $\mathcal{D} : \mathcal{L}^2 \to [0, \infty]$

(D1) $\mathcal{D}(C) = 0$ for constant random variables C

(D2) convexity, (D3) closedness

(D4) robustness: $\mathcal{D}(X) > 0$ for nonconstant X

Additional properties of major interest:

(D5) positive homogeneity: $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ when $\lambda > 0$

(D6) upper range boundedness: $\mathcal{D}(X) \leq \sup X - EX$

Note: (D1)+(D2) $\Longrightarrow \mathcal{D}(X + C) = \mathcal{D}(X)$ for constants C

symmetry not assumed, perhaps $\mathcal{D}(-X) \neq \mathcal{D}(X)$

Example: $\mathcal{D}(X) = \sigma(X) = ||X - EX||$ fails to satisfy (D6), but

$\mathcal{D}(X) = \sigma_+(X) = ||\max\{0, X - EX\}||$ satisfies all

Extended CAPM: obtained with $\mathcal{D}(X)$ replacing $\sigma(X)$ (finance)

Risk Versus Deviation

quantification of "cost/loss" versus quantification of uncertainty

Mean+deviation representation of risk measures

A **one-to-one** correspondence $\mathcal{D} \longleftrightarrow \mathcal{R}$ between regular risk measures $\mathcal{R}$ and regular deviation measures $\mathcal{D}$ is given by

$$\mathcal{R}(X) = EX + \mathcal{D}(X), \qquad \mathcal{D}(X) = \mathcal{R}(X - EX),$$

where moreover **monotonicity** (R6) of risk is characterized by

$$\mathcal{R}(X) \text{ satisfies (R6)} \quad \Longleftrightarrow \quad \mathcal{D}(X) \text{ satisfies (D6)}$$

Example 1: the risk measure $\mathcal{R}(X) = EX + \lambda\sigma(X)$, $\lambda > 0$, is regular but not monotonic because $\mathcal{D}(X) = \lambda\sigma(X)$ fails (D6)

Example 2: the deviation measure $\mathcal{D}(X) = \mathrm{CVaR}_p(X - EX)$ is not only regular but also, in addition, satisfies (D6)

Deviation From Error

looking now at a concept of "error" that can be asymmetric

Axioms for regular measures of error: $\mathcal{E} : \mathcal{L}^2 \to [0, \infty]$

(E1) $\mathcal{E}(0) = 0$, (E2) convexity, (E3) closedness

(E4) robustness: $\mathcal{E}(X) > 0$ for nonzero X

Additional properties of major interest:

(E5) positive homogeneity: $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$ when $\lambda > 0$

(E6) $\mathcal{E}(X) \leq |EX|$ when $X \leq 0$

Error projection (with respect to constants C)

For a regular error measure $\mathcal{E}$, let

$$\mathcal{D}(X) = \min_C \mathcal{E}(X - C), \quad \mathcal{S}(X) = \operatorname{argmin}_C \mathcal{E}(X - C).$$

Then $\mathcal{D}$ is a regular deviation measure, $\mathcal{S}$ the associated "statistic"

$$\mathcal{E}(X) \text{ satisfies (E6)} \quad \Longrightarrow \quad \mathcal{D}(X) \text{ satisfies (D6)}$$

$\longrightarrow$ $\mathcal{S}(X)$ is the constant C "nearest" to X with respect to $\mathcal{E}$

Some Error/Statistic Examples

Example 1: $\mathcal{E}(X) = \sqrt{E[X^2]}$ yields $\mathcal{S}(X) = EX$
this regular measure of error fails to satisfy (E6)

Example 2: $\mathcal{E}(X) = E|X|$ yields $\mathcal{S}(X) =$ median of X
this regular measure of error satisfies all

Example 3: $\mathcal{E}(X) = \sup|X|$ yields $\mathcal{S}(X) = \frac{1}{2}[\sup X + \inf X]$
this regular measure of error fails to satisfy (E6)

Example 4: $\mathcal{E}(X) = \frac{1}{1-p}E[\max\{0,X\}] - EX$ yields
the p-quantile statistic $\mathcal{S}(X) = q_p(X)$
this **asymmetric** regular measure of error satisfies all

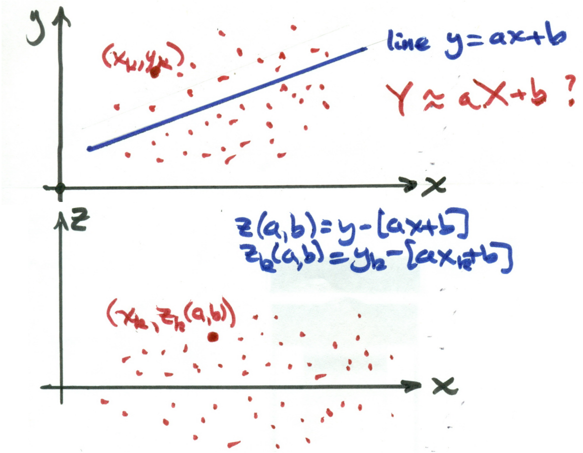
Example 5: $\mathcal{E}(X) = E[\exp X - X - 1]$ yields $\mathcal{S}(X) = \log E[\exp X]$
this **asymmetric** regular measure of error satisfies all

Databases — the Need for Estimation

Available information: e.g. a large collection of pairs (x_k, y_k)

Perspective: empirical distribution in x, y space of r.v.'s X, Y

Approximation: $Y \approx aX + b$, error gap $Z(a,b) = Y - [aX + b]$



Regression From a Broader Point of View

Y = random variable (scalar) to be understood in terms of
$X_1, \ldots, X_n$ = some "more basic" variables (e.g., "factors")

Approximation scheme: $Y \approx f(X_1, \ldots, X_n)$ for $f \in \mathcal{F}$

$\mathcal{F}$ = some specified class of functions $f : \boldsymbol{R}^n \to \boldsymbol{R}$

e.g. linear, $f(x_1, \ldots, x_n) = c_0 + c_1 x_1 + \cdots + c_n x_n$

error gap variable: $Z_f = Y - f(X_1, \ldots, X_n)$ for $f \in \mathcal{F}$

Regression problem, in general

minimize $\mathcal{E}(Z_f)$ over all $f \in \mathcal{F}$ for some error measure $\mathcal{E}$

Standard regression: $\mathcal{E}(Z_f) = (E[Z_f^2])^{1/2}$ "least squares"

Quantile regression: using $Z^+ = \max\{0, Z\}$, $Z^- = \max\{0, -Z\}$

$\mathcal{E}(Z_f) = E[\frac{p}{1-p} Z_f^+ + Z_f^-]$ at probability level $p \in (0,1)$

Koenker-Basset error, normalized

Effect of the Choice of Error Measure

error gap variable to be "made small": $Z_f = Y - f(X_1, \ldots, X_n)$

Regression problem "decomposition" (when $f \in \mathcal{F} \Rightarrow f + C \in \mathcal{F}$)

minimizing $\mathcal{E}(Z_f)$ over all $f \in \mathcal{F}$ corresponds to
minimizing $\mathcal{D}(Z_f)$ under the constraint $\mathcal{S}(Z_f) = 0$

Important issue for connecting with optimization:

parameterized "costs" $\underline{c}(x) = c(x, V)$ for $x = (x_1, \ldots, x_n)$
can be viewed as $Y = c(X, V)$ with $X = (X_1, \ldots, X_n)$
X = "randomized decision" tied to empirical sample at hand

- regression can serve then to get a "formula" for $\underline{c}(x) \approx f(x)$
- for using $\bar{c}(x) = \mathcal{R}(\underline{c}(x))$, shouldn't this be "tuned" to $\mathcal{R}$?
- $c(X, V)$ may only be supported by some (X, V) database!

Example: an Application to Composition of Alloys

Alloy model: a mixture of various metals

amounts of chief ingredients: $x = (x_1, \ldots, x_n)$ "design"

amounts of other ingredients: $v = (v_1, \ldots, v_m)$ "contaminants"

a "characteristic" to be controlled: y ideally kept ≤ 0, say

due to uncertainty, a quantile constraint may be envisioned

Background information: $y = c(x, v)$? no available formula!

there is only a database in (x, v, y)-space, $\{(x^k, v^k, y^k)\}_{k=1}^N$

- view the database as an empirical distribution for random variables $X = (X_1, \ldots, X_n)$, $V = (V_1, \ldots, V_m)$, Y
- use regression of Y on $X_1, \ldots, X_N$ to get a function $y = \bar{c}(x)$
- then impose the constraint $\bar{c}(x) \leq 0$ on the design x

shouldn't the regression adapt then to the intended constraint?

Some References on Generalized Regression

[1] R.T. Rockafellar, S. Uryasev, M. Zabarankin (2008), "Risk Tuning in Generalized Linear Regression," *Mathematics of Operations Research* 33, 712–729.

[2] R. T. Rockafellar, J. O. Royset (2015), "Measures of Residual Risk with Connections to Regression, Risk Tracking, Surrogate Models and Ambiguity," *SIAM Journal of Optimization* 25, 1179–1208.

Finishing the Quadrangle Scheme

$$\begin{array}{ccccc} & \text{risk } \mathcal{R} & \longleftrightarrow & \mathcal{D} \text{ deviation} & \\ \text{optimization} & \uparrow\downarrow & \mathcal{S} & \downarrow\uparrow & \text{statistics} \\ & \text{regret } \mathcal{V} & \longleftrightarrow & \mathcal{E} \text{ error} & \end{array}$$

Error versus regret

The one-to-one correspondence

$$\mathcal{E}(X) = \mathcal{V}(X) - EX, \qquad \mathcal{V}(X) = EX + \mathcal{E}(X)$$

coordinates error and regret with the same "statistic"

$$\mathcal{S}(X) = \operatorname*{argmin}_C \mathcal{E}(X - C) \longleftrightarrow \mathcal{S}(X) = \operatorname*{argmin}_C \{ C + \mathcal{V}(X - C) \}$$

Final links: nonunique but "natural" inversions $\mathcal{D} \to \mathcal{E}$, $\mathcal{R} \to \mathcal{V}$

The Mean-Based Quadrangle

articulated with a scaling parameter $\lambda > 0$

$$\mathcal{S}(X) = EX$$
$$= \text{mean}$$
$$\mathcal{E}(X) = \lambda(E[X^2])^{1/2}$$
$$= L^2\text{-error, scaled}$$
$$\mathcal{D}(X) = \lambda\,\sigma(X)$$
$$= \text{standard deviation, scaled}$$
$$\mathcal{R}(X) = EX + \lambda\,\sigma(X)$$
$$= \text{safety margin risk}$$
$$\mathcal{V}(X) = EX + \lambda(E[X^2])^{1/2}$$
$$= L^2\text{-regret}$$

properties: **aversity** with **convexity**, but NOT coherency

The Quantile-Based Quadrangle

at any probability level $p \in (0,1)$

$$\mathcal{S}(X) = q_p(X) = \mathrm{VaR}_p(X)$$
$$= \text{quantile}$$

$$\mathcal{R}(X) = Q_p(X) = \mathrm{CVaR}_p(X)$$
$$= \text{superquantile}$$

$$\mathcal{D}(X) = Q_p(X - EX) = \mathrm{CVaR}_p(X - EX)$$
$$= \text{superquantile deviation}$$

$$\mathcal{E}(X) = E[\tfrac{p}{1-p}X_+ + X_-]$$
$$= \text{Koenker-Basset error, normalized}$$

$$\mathcal{V}(X) = \tfrac{1}{1-p}E[X_+]$$
$$= \text{expected absolute loss, scaled}$$

properties: **aversity** with **coherency**

The Median-Based Quadrangle

the quantile case at probability level $p = 1/2$

$$\mathcal{S}(X) = \mathrm{VaR}_{1/2}(X) = q_{1/2}(X)$$
$$= \text{median}$$

$$\mathcal{R}(X) = \mathrm{CVaR}_{1/2}(X) = Q_{1/2}(X)$$
$$= \text{“supermedian” (average in tail above median)}$$

$$\mathcal{D}(X) = \mathrm{CVaR}_{1/2}(X - EX) = Q_{1/2}(X - EX)$$
$$= \text{supermedian deviation}$$

$$\mathcal{E}(X) = E|X|$$
$$= \mathcal{L}^1\text{-error}$$

$$\mathcal{V}(X) = 2E[X_+]$$
$$= \mathcal{L}^1\text{-regret}$$

properties: **aversity** with **coherency**

The Max-Based Quadrangle

corresponding to the limit of the quantile case as $p \to 1$

$$\mathcal{S}(X) = \tfrac{1}{2}[\sup X + \inf X]$$
$$= \text{center of the (essential) range of } X$$

$$\mathcal{R}(X) = \sup X$$
$$= \text{top of the (essential) range of } X$$

$$\mathcal{D}(X) = \tfrac{1}{2}[\sup X - \inf X]$$
$$= \text{radius of the (essential) range of } X$$

$$\mathcal{E}(X) = \sup |X|$$
$$= \mathcal{L}^\infty\text{-error}$$

$$\mathcal{V}(X) = \sup[X - EX]$$
$$= \mathcal{L}^\infty\text{-regret (max excess of “cost” over average)}$$

properties: **aversity** with **coherency**

The Log-Exponential-Based Quadrangle

$$\mathcal{S}(X) = \log E[\, e^X \,]$$
$=$ **dual** expression for Boltzmann-Shannon entropy

$$\mathcal{R}(X) = \log E[\, e^X \,]$$
$=$ yes, the same as $\mathcal{S}(X)$!

$$\mathcal{D}(X) = \log E[\, e^{X-EX} \,]$$
$=$ log-exponential deviation

$$\mathcal{E}(X) = E[\varepsilon(X)] \text{ with } \varepsilon(x) = e^x - x - 1 > 0 \text{ when } x \neq 0$$
$=$ exponential error

$$\mathcal{V}(X) = E[v(X)] \text{ with } v(x) = e^x - 1 \begin{cases} > 0 \text{ when } x > 0 \\ < 0 \text{ when } x < 0 \end{cases}$$
$=$ exponential regret

properties: **aversity** with **coherency**