Computational Project

SVAN 2016

Basic Course on Stochastic Programming, IMPA 2016

1 The Linear Stochastic Logistics Problem

This model describes a transportation network over which routing decisions have to be made at each stage. The random variable may be view as an uncertain traffic condition impacting the transportation costs on each route segment. The objective of the optimization is to minimize the total expected cost.

We start by presenting a 2-stage formulation. At the initial location A, one disposes of a total quantity of 1 cargo. A share of the cargo can be shipped to a northern warehouse W_n^1 with a known shipping cost of c_n^1 or to a southern warehouse W_s^2 with a known shipping cost of c_s^1; see Figure 1. The transport costs from those intermediate warehouses to the final destination, c_n^2, c_s^2 are subject to uncertain traffic conditions, given by ξ. The random variable ξ can take two values, depending on whether there is light traffic ($\xi = 0$) or heavy traffic ($\xi = 1$). The traffic information becomes available only when the cargo arrives at the intermediate warehouses. At this point, the decision maker may opt to transfer a certain share of the goods from one warehouses to another, at a cost r_n^2 (relocating the goods to the northern warehouse) or r_s^2 (relocating the goods to the southern warehouse). This relocation and the subsequent shipment to location B are the recourse decisions and are made after the event ξ is known. The relocation costs are known with certainty and do not depend on ξ. Figure 1 illustrates the network for the two-stage problem. At time stage t, the first two components of the decision vector $x_t = (x_t^1, x_t^2, x_t^3, x_t^4)^\top$ represents the amount of cargo shipped to the northern and southern warehouses while the last two correspond to the cargo reshipped from the north to the south and visceversa. With this notaiton, the described system can modeled as

$$\min_{x_1,x_2} c_1^\top x_1 + \mathbb{E}\Big[c_2(\xi)^\top x_2\Big], \tag{1a}$$

under the conservation constraints

$$\begin{aligned} A_1 x_1 &= b_1, & (1b)\\ B_2 x_1 + A_2 x_2 &= b_2, & (1c) \end{aligned}$$

and the capacity constraints

$$\begin{aligned} x_1 &\in [0,1]^4, & (1d)\\ x_2 &\in [0,1]^4. & (1e) \end{aligned}$$

The matrices in the conservation constraints are

$$\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}, \\
B_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}, \\
b_1 &= 1, \\
b_2 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

and the cost vectors are

$$\begin{aligned}
c_1 &= \begin{pmatrix} c_n^1 & c_s^1 & 0 & 0 \end{pmatrix}^\top \\
c_2(\xi) &= \begin{pmatrix} c_n^2(\xi) & c_s^2(\xi) & r_s^2 & r_n^2 \end{pmatrix}^\top.
\end{aligned}$$

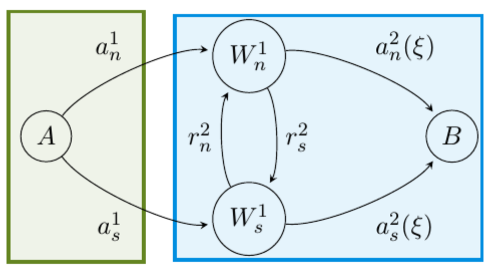

Figure 1: 2-stage stochastic logistics model illustration. The objective is to transport a cargo from node A to node B, shipping the cargo, at stage t, between warehouses located in the north W_n^t or south W_s^t. The segments are named after *advance* cargo via north at stage t (c_n^t), *advance* cargo via south at stage t (c_s^t), *relocate* cargo south at stage t (r_s^t), *relocate* cargo north at stage t (r_n^t). The costs of advancing the cargo in stage 2 are impacted by uncertain traffic conditions modeled by the random variable ξ.

A multi-stage logistics model with an adjustable time horizon, T, is similar to the 2-stage case. The information of the process $\xi = (\xi_2, \cdots, \xi_T)$ enters the system sequentially before the recourse decision is made. The model associated to this case is an immediate extension of problem (1). Figure 2 illustrates the system.

In our simulations, we will use the instance

- $c_n^1 = 3$, $c_s^1 = 2$,
- $r_s^t = 1$ and $r_n^t = 2$ for all $t = 2, \cdots, T$,
- $c_n^t = 3\chi_{\xi_t=0}(\xi_t) + 4\chi_{\xi_t=1}(\xi_t)$ for all $t = 2, \cdots, T$, and

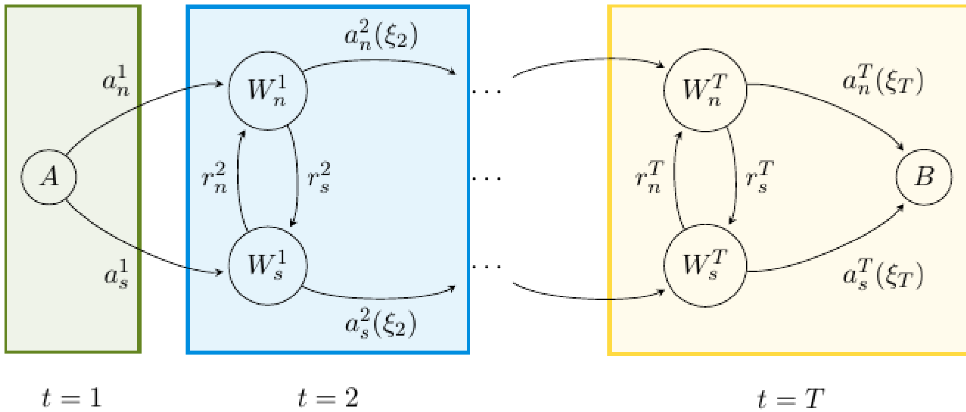

Figure 2: Multi-stage stochastic logistics system illustration.

- $c_s^t = 2\chi_{\{\xi_t=0\}}(\xi_t) + 15\chi_{\{\xi_t=1\}}(\xi_t)$ for all $t = 2, \cdots, T$ where, for a set G, $\chi_G(\xi)$ is equal to 1 for $\xi \in G$ and 0 otherwise.

In what follows, we consider an SAA approximation of the problem, with N equiprobable scenarios.

2 The project

The work is divided in three parts, described below.

2.1 Two stage model

The file `ScenLinLog.mat` downloadable from the course webpage contains a matrix with (i, j) element denoted by σ_{ij}. For $T = 2$, we use those elements to generate five groups of N values of $\xi \in \{0, 1\}$, as follows.

$$\text{For } j = 1, \ldots, 5 \text{ the group } j \text{ consists of } \{\xi_j^1, \ldots, \xi_j^N\} \text{ such that } \quad \xi_j^n = \begin{cases} 1 & \text{if } \sigma_{jn} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Code a black-box `BBCut` for (1), following the structure of the computational list. Then, for $N \in \{10^2, 10^3\}$, solve five instances of (1) using the methods below

1. the L-shaped method implemented in the computational list,
2. the stochastic decomposition implemented in the computational list,
3. a proximal bundle method, described below.

For each of the methods above, and for $N = 10^2$ and $N = 10^3$, report your resuls in Table 1.

j	L-shaped # Call	CPU (s)	f_N^*	StocDec # Call	CPU (s)	f_N^*	Proxbun # Call	CPU (s)	f_N^*
1									
2									
3									
4									
5									
Mean									
Std. Dev.									

Table 1: Results from part I

Proximal bundle method

This is a regularized L-shaped algorithm, which works with the index sets $\mathbb{O}_k$ and $\mathbb{F}_k$ in $\{1, 2, \ldots, k\}$.

Given a parameter $\mu_k > 0$ and a center $\hat{x}^k$, the proximal bundle master subproblem is

$$\begin{cases} \min\limits_{x,r} & c^\top x + r + \frac{1}{2}\mu_k \|x - \hat{x}^k\|^2 \\ \text{s.t.} & Ax = b \\ & \alpha^j + {\beta^j}^\top x \le r, & \forall\, j \in \mathbb{O}_k \\ & \tilde{\alpha}^j + {\tilde{\beta^j}}^\top x \le 0, & \forall\, j \in \mathbb{F}_k \\ & r \in \mathbb{R},\ 0 \le x_i & \forall\, i = 1, \ldots, n_1, \end{cases} \tag{3}$$

Code in MATLAB the proximal bundle method:

Proximal bundle method

- **Step 0: initialization**. Choose a tolerance `Tol` > 0, the maximum number of iterations `MaxIt` > 0, a descent parameter $m \in (0, 1)$, and a feasible first-stage point $x^1 \in \{x \in \mathbb{R}_+^{n_1} : Ax = b\}$. Define $k = 1$, $f_0^{\text{up}} = \infty$ and $\mathbb{A}_0 = \mathbb{F}_0 = \emptyset$. Let $\hat{x}^1 = x^1$, $\Delta_0 = -\infty$, and $\mu_0 = \dfrac{0.5|c|^2}{1 + |c^\top x^0|}$.

- **Step 1: oracle call**. Call the black-box `BBCut` at x^k.
  - If `pars.ind` $= 0$, define $\mathbb{O}_k = \mathbb{A}_{k-1} \cup \{k\}$ and $\mathbb{F}_k = \mathbb{F}_{k-1}$.

    Let $f_k = c^\top x^k +$ `pars.Q` and $\mu_k^{\text{aux}} = 0.5\mu_{k-1}\left(1 + \dfrac{f_{k-1}^{\text{up}} - f_k}{\Delta_{k-1}}\right)^{-1}$.

    If $f_k \le f_{k-1}^{\text{up}} - m\Delta_{k-1}$, update: $\hat{x}^k = x^k$, $f_k^{\text{up}} = c^\top x^k +$ `pars.Q`, and $\mu_k = \max\Big(0.1\mu_{k-1}, \mu_k^{\text{aux}}\Big)$.

    Otherwise, take $\hat{x}^k = \hat{x}^{k-1}$, $f_k^{\text{up}} = f_{k-1}^{\text{up}}$, and $\mu_k = \max\Big(\mu_{k-1}, \min(1.1\mu_{k-1}, \mu_k^{\text{aux}}, 10^8)\Big)$.
  - If `pars.ind` > 0, define $\mathbb{O}_k = \mathbb{O}_{k-1}$ and $\mathbb{F}_k = \mathbb{F}_{k-1} \cup \{k\}$. Set $f_k^{\text{up}} = f_{k-1}^{\text{up}}$.
  - If `pars.ind` $= -1$, stop. Error in the black-box.

- **Step 2: next iterate**. Compute (x^{k+1}, r^{k+1}) by solving (3). Define $\mathbb{A}_k = \{j \in \mathbb{O}_k : \alpha^j + {\beta^j}^\top x^{k+1} = r^{k+1}\}$.

- **Step 3: stopping test**. Set $\Delta_k = f_k^{\text{up}} - (c^\top x^{k+1} + r^{k+1})$. If $\Delta_k \le (1 + |f_k^{\text{up}}|)$`Tol`, stop. Return $\hat{x}^k$ and f_k^{up}.

- **Step 4: loop**. If $k =$ `MaxIt`, stop: maximum number of iterations is reached. Return the best candidate $\hat{x}^k$ and its functional value f_k^{up}. Otherwise, set $k = k + 1$ and go back to Step 1.

In your experiments, define x^1 as a solution to the first-stage LP: $\min\ c^\top x \quad \text{s.t.} \quad Ax = b,\ \ x \ge 0$.

2.2 Multi-stage linear logistic problem

To generate T-dimensional scenarios, we proceed along the lines of (2):

$$\text{for } n = 1, \ldots, N, \text{and } t = 1, \ldots, T \text{ the scenario } n \text{ is } (\xi_1^n, \ldots, \xi_T^n) \text{ with } \quad \xi_t^n = \begin{cases} 1 & \text{if } \sigma_{tn} > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $T \in \{3, 4, 5\}$, write the multistage linear program and solve the deterministic equivalent problem with $N = 10$ (only one instance). Report the obtained values in Table 2.

T	# Call	CPU (s)	f_N^*
3			
4			
5			

Table 2: Results from part II

2.3 PhD students

Depending on your background, choose one of the two following subjects on 2-stage stochastic linear programs with fixed recourse to develop as additional work.

2.3.1 Convergence proof of the stochastic decomposition method

Suggested reference: Stochastic decomposition: An algorithm for two-stage linear programs with recourse. JL Higle, S Sen. Mathematics of operations research 16 (3), 650-669, 1991.
Work: write down a self contained convergence proof for this method.

2.3.2 Implementation of an inexact bundle method

Suggested reference: Inexact bundle methods for two-stage stochastic programming. SIAM Journal on Optimization, v. 21, p. 517-544, 2011. W. de Oliveira, C. Sagastizábal, and S. Scheimberg.
Work: apply the method to the linear logistic two-stage problem and to the example of the computational list.

Final remarks

Each student should send to `svanprog@impa.br` before July 4th, 2016:

- a report (in pdf format) on this work, containing answers to the questions and all the MATLAB sources in a readable manner.
- all the MATLAB sources (in .m format) and a main script called `BAS-final` gathering all your solvers, following the guidelines of the compuational list.