Handling Uncertainty in Stochastic Programming

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with help from

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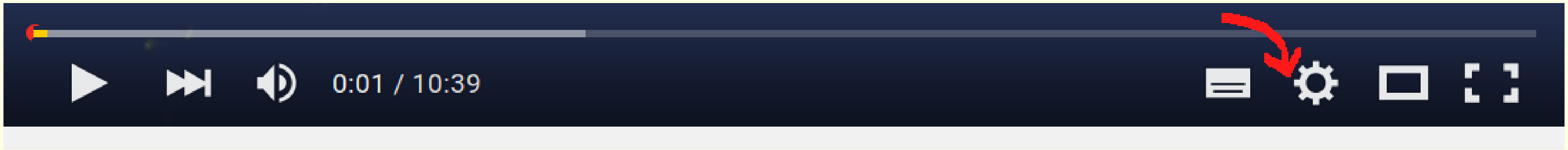

for best viewing

Here-and-now variables

- Asking feasibility for any future realization of uncertainty can be too restrictive.
- Depending on the data, extreme rare events may exist, and they can make the set of a.s. feasible points empty.

Two alternatives:

- To define a model for which extreme events do not constrain “too much” the a.s. feasible points.

  A model with recourse

- To consider points that are feasible with only some probability level.

  A model with chance constraints

Models with recourse

In the oil production problem, suppose the refinery agrees with the distribution company in paying a penalty if there is a shortage in production.

The penalty is the price the distribution company pays if it has to buy the missing gasoline directly in the market.

The amount of the penalty can be determined **after** observing the realization of uncertainty (the eventual production of gas of the refinery and the level of demand from the clients).

The corresponding recourse function measures the constraint violation, and it is a function of wait-and-see variables.

The production plan of the refinery is still a here-and-now variable.

Models with recourse

Consider the random vector $\xi = (\eta_1, \eta_2, \zeta_1, \zeta_2)$ and the random variables

- $h_1(\xi) = 180 + \zeta_1$ and $h_2(\xi) = 162 + \zeta_2$, (demand)
- $\alpha(\xi) = 2 + \eta_1$ and $\beta(\xi) = 3.4 - \eta_2$, (productivity)

The constraints, for an event ξ, have the expression

$$\left\{ \begin{array}{rrll} x_1 & +x_2 & \leq 100 & \\ \alpha(\xi)x_1 & +6x_2 & \geq h_1(\xi) & \text{(demand)} \\ 3x_1 & +\beta(\xi)x_2 & \geq h_2(\xi) & \text{(demand)} \\ & x_1, x_2 & \geq 0 & \end{array} \right.$$

Lack of feasibility comes from demand satisfaction constraints

Models with recourse

Instead of asking feasibility for all events, we can allow shortages, charging a penalty for constraint violation

For each event ξ we introduce two slack variables, $y_1(\xi)$ and $y_2(\xi)$, representing unsatisfied demand. Constraint set:

$$\begin{cases} \begin{array}{rrrl} x_1 & +x_2 & & \leq 100 \\ \alpha(\xi)x_1 & +6x_2 & +y_1(\xi) & \geq h_1(\xi) \\ 3x_1 & +\beta(\xi)x_2 & +y_2(\xi) & \geq h_2(\xi) \\ & x_1, x_2 \geq 0, & y_1(\xi), y_2(\xi) & \geq 0 \end{array} \end{cases}$$

Remark: The variables $y_1(\xi)$ and $y_2(\xi)$ are of the wait-and-see type.

Models with recourse

$$\begin{cases} x_1 & +x_2 & & \leq 100 \\ \alpha(\xi)x_1 & +6x_2 & +y_1(\xi) & \geq h_1(\xi) \\ 3x_1 & +\beta(\xi)x_2 & +y_2(\xi) & \geq h_2(\xi) \\ & x_1, x_2 \geq 0, & y_1(\xi), y_2(\xi) & \geq 0 \end{cases}$$

For any point (x_1, x_2) that satisfies the deterministic capacity constraint, there are wait-and-see variables that make the point feasible for the demand constraint. So asking for a.s. feasible points is not a restrictive condition[a]

[a] a.s. stands for almost sure: the relation holds for all ξ, except for a set of null measure

Models with recourse

Suppose shortage cost is $7y_1(\xi) + 12y_2(\xi)$ and consider a fixed event $\tilde{\xi}$. The (deterministic) LP is

$$
\left\{
\begin{array}{lrrrl}
\min & 2x_1 + 3x_2 + 7y_1(\tilde{\xi}) + 12y_2(\tilde{\xi}) & & & \\
 & x_1 & +x_2 & & \leq 100 \\
 & \alpha(\tilde{\xi})x_1 & +6x_2 & +y_1(\tilde{\xi}) & \geq h_1(\tilde{\xi}) \\
 & 3x_1 & +\beta(\tilde{\xi})x_2 & +y_2(\tilde{\xi}) & \geq h_2(\tilde{\xi}) \\
 & & x_1, x_2 \geq 0, & y_1(\tilde{\xi}), y_2(\tilde{\xi}) & \geq 0
\end{array}
\right.
$$

Now, among all feasible points (x_1, x_2), we need to choose one, before any future event takes place, that is satisfactory according to some criterion.

Models with recourse

Question: having chosen the values (x_1, x_2), what can I expect in future when the uncertain event takes place?

Models with recourse

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For **one** future event $\tilde{\xi}$, given the decision (x_1, x_2),

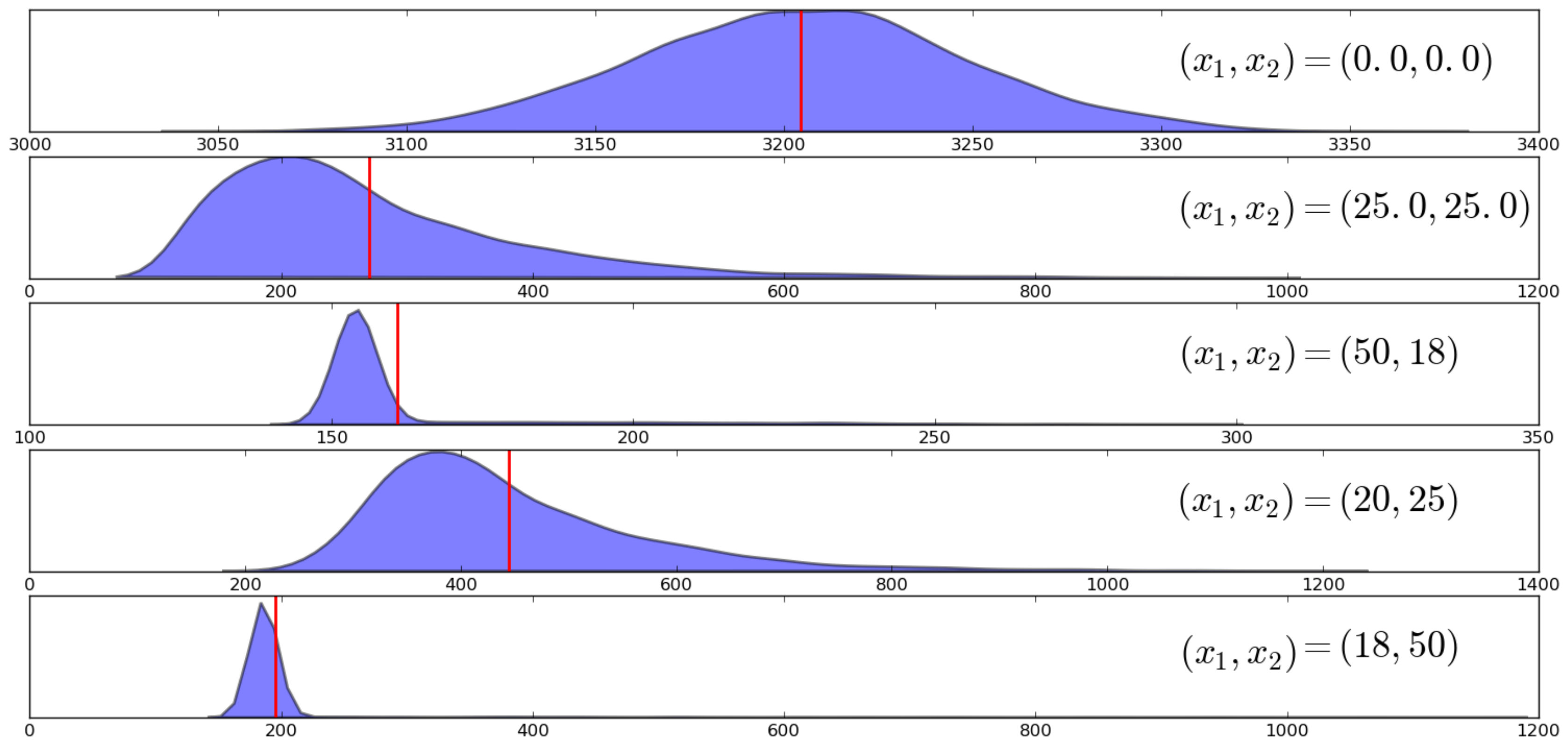
$$C(x_1, x_2, \tilde{\xi}) := \left\{ \begin{array}{lrrrl} \min & 2x_1 + 3x_2 + 7y_1(\tilde{\xi}) + 12y_2(\tilde{\xi}) & & & \\ & x_1 & +x_2 & & \leq 100 \\ & \alpha(\tilde{\xi})x_1 & +6x_2 & +y_1(\tilde{\xi}) & \geq h_1(\tilde{\xi}) \\ & 3x_1 & +\beta(\tilde{\xi})x_2 & +y_2(\tilde{\xi}) & \geq h_2(\tilde{\xi}) \\ & & x_1, x_2 \geq 0, & y_1(\tilde{\xi}), y_2(\tilde{\xi}) & \geq 0 \end{array} \right.$$

is the optimal cost, a random variable if we let ξ vary.

We need to encompass all possible realizations of ξ

Models with recourse

Given (x_1, x_2), as a function of ξ, $C(x_1, x_2, \xi)$ is random and describes the optimal cost associated with realizations of ξ.



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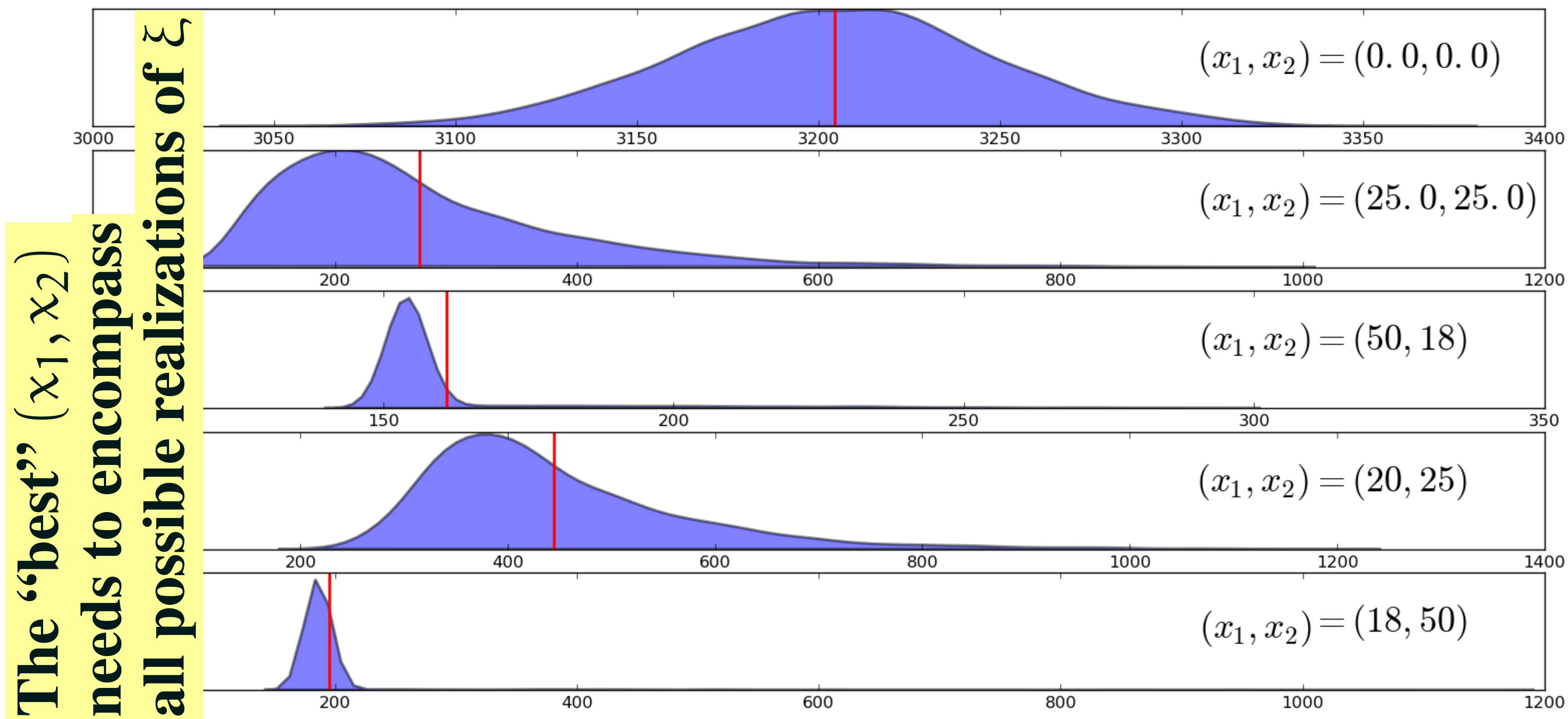

Models with recourse

One criterion for choosing the "best" (x_1, x_2) is to use a "functional"

$$\mathcal{M} : \mathcal{Z} \to \mathbb{R}$$

that measures the "goodness of random variables" according to our perception (conservative, risk neutral, risk averse)

The subspace $\mathcal{Z}$ is such that $C(x_1, x_2, \cdot) \in \mathcal{Z}$ for all x_1, x_2,

The "best" (x_1, x_2) is $(\bar{x}_1, \bar{x}_2)$ solving

$$\min_{x_1, x_2} \mathcal{M}\Big[C(x_1, x_2, \cdot)\Big]$$

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An important example is to choose $\mathcal{M} = \mathbb{E}$, an appropriate measure for repetitive events (criterion neutral w.r.t. to risk)

Models with recourse

It turns out that $\min_{x_1,x_2} \mathbb{E}\Big[C(x_1,x_2,\cdot)\Big]$ is equivalent to

$$\left\{\begin{array}{llllll} \min & 2x_1 + 3x_2 + \mathbb{E}[7y_1(\xi) + 12y_2(\xi)] & & & & \\ & x_1 & +x_2 & & \le 100 & \\ & \alpha(\xi)x_1 & +6x_2 & +y_1(\xi) & \ge h_1(\xi) & \\ & 3x_1 & +\beta(\xi)x_2 & +y_2(\xi) & \ge h_2(\xi) & \\ & & x_1,x_2 \ge 0, & y_1(\xi), y_2(\xi) & \ge 0 & \end{array}\right. \left.\vphantom{\begin{array}{l} a\\ a\\ a\\ a \end{array}}\right\} \text{a.s.}[\xi]$$

For **general** distributions of ξ we have that

- Computing $\mathbb{E}\Big[7y_1(\xi) + 12y_2(\xi)\Big]$ can be very expensive.
- There may be **infinitely many** constraints.

Models with recourse

In (very) important special cases, ξ has a finite support: $\xi \in \{\xi_1, \xi_2, \ldots, \xi_K\}$, where ξ_i has probability p_i.

$$\left\{\begin{array}{lrrrl} \min & 2x_1 + 3x_2 + \sum_{i=1}^{K} p_i[7y_1(\xi_i) + 12y_2(\xi_i)] & & & \\ & x_1 & +x_2 & & \leq 100 \\ & \alpha(\xi_i)x_1 & +6x_2 & +y_1(\xi_i) & \geq h_1(\xi_i) \\ & 3x_1 & +\beta(\xi_i)x_2 & +y_2(\xi_i) & \geq h_2(\xi_i) \\ & & x_1, x_2 \geq 0, & y_1(\xi_i), y_2(\xi_i) & \geq 0 \end{array}\right. \quad i = 1, \ldots, K$$

The number wait-and-see variables and constraints can grow fast with the number of scenarios. The set $\{\xi_1, \ldots, \xi_K\}$ can be a discrete version of a continuous ξ.

The impact of scenario representation: dimensionality explosion

Consider uncertainties only on the right-hand-side (RHS):

$$\left\{\begin{array}{lllllll}
\min & 2x_1 + 3x_2 + \sum_{i=1}^{K} p_i [7y_1(\xi_i) + 12y_2(\xi_i)] & & & & & \\
& x_1 & +x_2 & & \leq 100 & & \\
& x_1 & +6x_2 & +y_1(\xi_i) & \geq h_1(\xi_i), & & i = 1,\ldots,K \\
& 3x_1 & +x_2 & +y_2(\xi_i) & \geq h_2(\xi_i), & & i = 1,\ldots,K \\
& & x_1, x_2 \geq 0, & y_1(\xi_i), y_2(\xi_i) & \geq 0, & & i = 1,\ldots,K
\end{array}\right.$$

- Fifteen realizations were drawn for $h_1(\xi)$ and $h_2(\xi)$.
- A total of $K = 15^2 = 225$ scenarios were considered.
- We obtain an LP with $2 + 225 * 2 = 552$ variables.

The impact of scenario representation: dimensionality explosion

From a toy LP with 2 variables we got a SLP with 552 variables after a quite crude approximation of uncertainty

$$
\left\{
\begin{array}{lllll}
\min & 2x_1 + 3x_2 + \sum_{i=1}^{K} p_i [7y_1(\xi_i) + 12y_2(\xi_i)] & & & \\
 & x_1 \quad +x_2 & & \leq 100 & \\
 & x_1 \quad +6x_2 & +y_1(\xi_i) & \geq h_1(\xi_i), & i = 1, \ldots, K \\
 & 3x_1 \quad +x_2 & +y_2(\xi_i) & \geq h_2(\xi_i), & i = 1, \ldots, K \\
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\end{array}
\right.
$$

- **Fifteen realizations** were drawn for $h_1(\xi)$ and $h_2(\xi)$.
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The impact of scenario representation:

reliability

$$\begin{cases} \min & 2x_1 + 3x_2 + \sum_{i=1}^{K} p_i [7y_1(\xi_i) + 12y_2(\xi_i)] & & & \\ & x_1 \quad +x_2 & & \leq 100 & \\ & x_1 \quad +6x_2 & +y_1(\xi_i) & \geq h_1(\xi_i), & i = 1,\ldots,225 \\ & 3x_1 \quad +x_2 & +y_2(\xi_i) & \geq h_2(\xi_i), & i = 1,\ldots,225 \\ & x_1, x_2 \geq 0, & y_1(\xi_i), y_2(\xi_i) & \geq 0, & i = 1,\ldots,225 \end{cases}$$

An optimal solution for this LP is $\bar{x}^{225} = (38.539, 20.539)$.
Its optimal value is $C(\bar{x}^{225}) = 140.747$ and the cost of producing $\bar{x}^{225}$ is $C_1 = f(\bar{x}^{225}) = 138.694$.

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$$\left\{\begin{array}{llllll} \min & 2x_1+3x_2+\sum_{i=1}^{K} p_i[7y_1(\xi_i)+12y_2(\xi_i)] & & & & \\ & x_1 & +x_2 & & \leq 100 & \\ & x_1 & +6x_2 & +y_1(\xi_i) & \geq h_1(\xi_i), & i=1,\ldots,225 \\ & 3x_1 & +x_2 & +y_2(\xi_i) & \geq h_2(\xi_i), & i=1,\ldots,225 \\ & & x_1,x_2 \geq 0, & y_1(\xi_i),y_2(\xi_i) & \geq 0, & i=1,\ldots,225 \end{array}\right.$$

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This is a feasible point for the above LP.

The impact of scenario represe

reliability

What is the probability for $\bar{x}^{225}$ to be feasible for the "true problem" (ξ continuous)?

$$
\begin{cases}
\min & 2x_1 + 3x_2 + \sum_{i=1}^{K} p_i[7y_1(\xi_i) + 12y_2(\xi_i)] & & \\
& x_1 \quad +x_2 & \leq 100 & \\
& x_1 \quad +6x_2 \quad +y_1(\xi_i) & \geq h_1(\xi_i), & i = 1,\dots,225 \\
& 3x_1 \quad +x_2 \quad +y_2(\xi_i) & \geq h_2(\xi_i), & i = 1,\dots,225 \\
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\end{cases}
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The impact of scenario representation:

reliability

Assuming the random vector $h(\xi)$ is normally distributed, the answer can be obtained by employing the Matlab function `mvncdf`.

$$\mathbb{P}\left[\begin{array}{lcl} 2\,\bar{x}_1^{225} + 6\,\bar{x}_2^{225} & \geq & h_1(\xi) \\ 3\,\bar{x}_1^{225} + 3\,\bar{x}_2^{225} & \geq & h_2(\xi) \end{array}\right] = 0.912$$

The Matlab command is:

$$\mathtt{mvncdf}([2\ 6; 3\ 3] * \bar{x}, [180; 162], [12^2\ 0; 0\ 9^2])$$

The impact of scenario representation:

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$$\mathbb{P}\left[\begin{array}{rcl} 2\bar{x}_1^{225}+6\bar{x}_2^{225} & \geq & h_1(\xi) \\ 3\bar{x}_1^{225}+3\bar{x}_2^{225} & \geq & h_2(\xi) \end{array}\right]=0.912$$

The Matlab command is:

$$\texttt{mvncdf}([2\ 6;3\ 3]*\bar{x},[180;162],[12^2\ 0;0\ 9^2])$$

We can do the same for the deterministic (no ξ), and the worst-case solutions.

Assessing quality of the solution

Model	$\bar{x}_1$	$\bar{x}_2$	$f(\bar{x})$	Feasibility
Deterministic	36	18	126	0.250
Worst-case	48.018	25.548	172.681	1
Recourse	38.539	20.539	138.694	0.912

- Deterministic: gives the cheapest and less reliable decision, it has only 25% of chance to be feasible. (This makes sense: uncertainty was simply ignored!)
- Worst-case is expensive (172.681) but feasible for all future realization
- Recourse: decision not too expensive, with reasonable probability of being feasible. Reliability can only be incresed at the expense of increasing the number of scenarios.

Limitations of models with recourse

A **stochastic program with recourse** seems a good model for the oil production example.

In this problem, “recourse” is always a possibility: if shortage happens, the company can buy (at more expensive prices) crude oil in the market to produce gasoline to meet the demand. It can even buy gasoline, in fact.

However, there are applications in which such “recourse” simply does not exist!

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However, there are applications in which such “recourse” simply does not exist!

- **There is no recourse for lost lives!**
  (Think of disasters caused by wrongly planned dams.)

Probabilistic models

When feasibility (regarded as safety) plays a role sometimes more important than optimality, stochastic programs with recourse are not even a choice.

- A formulation with probability constraints (chance-constrained programs) is a more suitable model.

Let's make the (non-realistic) assumption that there is no recourse in our example: it means that shortage cannot be covered by immediate market purchases.

- Given this premise, the aim is to find the best decision that is at least 95% feasible for all possible realizations of the random variables $h_1(\xi)$ and $h_2(\xi)$.

LP with joint probabilistic constraints

$$\left\{\begin{array}{ll} \min & 2x_1 + 3x_2 \\ & x_1 + x_2 \leq 100 \\ & \mathbb{P}\left[\begin{array}{lcl} 2x_1 + 6x_2 & \geq & h_1(\xi) \\ 3x_1 + 3x_2 & \geq & h_2(\xi) \end{array}\right] \geq 0.95 \\ & x_1, x_2 \geq 0. \end{array}\right.$$

This problem is tractable thanks to the assumption on the probability distribution of $h(\xi)$.

LP with joint probabilistic constraints

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This problem is tractable thanks to the assumption on the probability distribution of $h(\xi)$.

Solving our chance-constrained program by a specialized method gives $\bar{x}^{\mathbb{P}} = (37.758, 21698)$, with $f(\bar{x}^{\mathbb{P}}) = 140.616$, noting that $\bar{x}^{\mathbb{P}}$ has 95% of chances of being feasible.

Assessing quality of the solution

Our table then becomes:

Model	$\bar{x}_1$	$\bar{x}_2$	$f(\bar{x})$	Feasibility
Deterministic	36	18	126	0.250
Worst-case	48.018	25.548	172.681	1
Recourse	38.539	20.539	138.694	0.912
Chance-Constraints	37.758	21.698	140.616	0.950

Note that the cost $f(\bar{x})$ is slightly increased compared with the Deterministic solution if we observe the drastic increase of reliability.

Probabilistic models, or chance-constrained programs

$$\begin{cases} \min & f(x) \\ \text{s.t.} & x \in X \\ & \mathbb{P}[g(x) \geq \xi] \geq p\,. \end{cases}$$

In general very challenging optimization problems:

- computing numerically $\mathbb{P}$ is only possible for a few classes of probability distributions
- the feasible set is in general non convex
- when the random variable is discrete the problem becomes a large scale mixed-integer programming problem, exceeding capability of standard MINLP solvers

Probabilistic models, or chance-constrained programs

Like for SP with recourse, solving chance-constrained problems requires specialized methods

$$\begin{cases} \min & f(x) \\ \text{s.t.} & x \in X \\ & \mathbb{P}[g(x) \geq \xi] \geq p\,. \end{cases}$$

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Homework

For the oil production example, considering RHS uncertainty only of the form

- $h_1 = 180 + \zeta_1$ com $\zeta_1 \sim \mathcal{N}(0, 16)$.
- $h_2 = 163 + \zeta_2$ com $\zeta_2 \sim \mathcal{N}(0, 9)$.

use Matlab/Octave to generate K demand scenarios and solve the models

1. Deterministic
2. Worst-case
3. Scenario Analysis
4. Chance-constrained

5. With recourse, considering that the gasoline bought in case of shortage costs 7 and 12.

Try your models first with $K = 1$ and then run the code with $K = 10$ and $K = 100$ scenarios.

Compare the different optimal values and solutions found.

To assess the quality solution, compute the reliability of each $\bar{x}$, using the Matlab function `mvncdf`.