TWO-STAGE SLP
EXPECTED RECOURSE FUNCTION AND OPTIMALITY CONDITIONS

Claudia Sagastizábal

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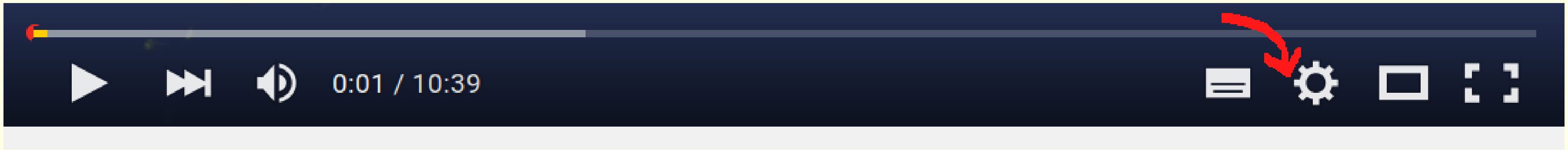

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Measurability in an abstract space Ω

$\mathcal{F}$ is a class of measurable sets F in Ω with probability measure P

- $F \in \mathcal{F}$ implies $\Omega \backslash F \in \mathcal{F}$ $(\Omega \in \mathcal{F})$
- $F_i \in \mathcal{F}$ for $i = 1, \ldots$ with $F_i \cap F_j = \emptyset$ implies $P(\cup_i A_i) = \sum_i P(A_i)$

The sets F are the events, and the triplet $(\Omega, \mathcal{F}, P)$ defines a probability space

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The class $\mathcal{F}$ is called a **σ-algebra**

Illustration

Consider the SLP in $\mathbb{R}$ and its recourse problem

$$\begin{cases} \min & cx \\ \text{s.t.} & x = \omega \text{ a.e.} \\ & x \geq 0 \end{cases} \qquad Q(x,\omega) = \begin{cases} \min & q^+ y^+ + q^- y^- \\ \text{s.t.} & y^+ + y^- = \omega - x \\ & y^+, y^- \geq 0 \end{cases}$$

The 2SLP is

$$\begin{cases} \min & cx + \mathbb{E}[Q(x,\omega)] \\ \text{s.t.} & x \geq 0 \end{cases}$$

– $\Pi(q) \neq \emptyset \iff q^+ + q^-$ (sufficiently expensive recourse)

– when $q^+ = q^- = 1$, $Q(x,\omega) = |\omega - x|$

– Ω with finite support, the function $\phi(x) := \mathbb{E}[Q(x,\omega)]$ is convex and polyhedral, with kinks $\omega \in \Omega$

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– Ω with finite support, the function $\phi(x) := \mathbb{E}[Q(x,\omega)]$ is convex and polyhedral, with kinks $\omega \in \Omega$ **and for a continuous distribution?**

Illustration: the setting

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with Ω continuous, so that

$$\phi(x) = \mathbb{E}[Q(x,\omega)] = \int_{-\infty}^{+\infty} Q(x,\omega)\, d\mathbb{P}(\omega)$$

Let F and $\bar{F}$ denote the cdf and the tail of a random variable $\xi : \Omega \to \mathbb{R}$:

$$F(z) = \mathbb{P}(\xi(\omega) \leq z) \quad \text{and} \quad \bar{F}(z) = \mathbb{P}(\xi(\omega) \geq z)\,.$$

$$F(z) = \begin{cases} 0 \\ \frac{x-a}{b-a} \\ 1 \end{cases} \quad \text{and} \quad \bar{F}(z) = \begin{cases} 1 \\ \frac{b-x}{b-a} \\ 0 \end{cases}$$

Take $q^+ = q^- = 1$ so that $Q(x,\omega) = |\omega - x|$, and suppose $\omega \sim U[a,b]$ with $0 \leq a < b$. We (you) will show that ϕ is the **smooth function** $\phi(x_0) = \frac{x_0^2 + (x_0 - a)^2}{2(b-a)}$ for $x_0 \in (a,b)$.

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$$\mathbf{F}(\mathbf{z}) = \begin{cases} 0 & z \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & z \ge b \end{cases} \quad \text{and} \quad \bar{\mathbf{F}}(\mathbf{z}) = \begin{cases} 1 & z \le a \\ \frac{b-x}{b-a} & a < x < b \\ 0 & z \ge b \end{cases}$$

Take $q^+ = q^- = 1$ so that $Q(x,\omega) = |\omega - x|$, and suppose $\omega \sim U[a,b]$ with $0 \le a < b$. We (you) will show that ϕ is the **smooth function** $\phi(x_0) = \frac{x_0^2 + (x_0 - a)^2}{2(b-a)}$ for $x_0 \in (a,b)$.

Illustration: proof steps (your turn!)

- Result (show it): for a random variable $\xi = \xi(\omega) \geq 0$

$$\mathbb{E}[\xi] = \int_0^{+\infty} \bar{F}(z) dz.$$

- $Q(x_0, \omega) = \xi_+ + \xi_-$ for $\xi_+ = (Q(x_0, \omega))_+$ and $\xi_- = (Q(x_0, \omega))_-$

- Result can be applied to each term to prove that (do it)
  - $\mathbb{E}[\xi_+] = \int_{x_0}^{b} \bar{F}(z) dz$
  - $\mathbb{E}[\xi_-] = \int_{a}^{x_0} F(z) dz$

  (hint: for all $z \geq 0$, $\xi_+ \geq z \iff \xi \geq z$)

- Compute the integrals above.

(necessary and sufficient) Optimality Condition

For the 2SLP and uncertainty with finite support, $\Omega = \{\omega^1, \ldots, \omega^K\}$:

$$\min_{\mathbf{x} \in \mathbf{X}} \mathbf{c}^\top \mathbf{x} + \phi(\mathbf{x}) \qquad \text{for} \qquad \mathbf{X} := \{\mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} = \mathbf{b}\},$$

$b \in \mathbb{R}^{nb}$

where $\phi(x) = \mathbb{E}\left[Q(x, \xi)\right] = \sum_{k=1}^{K} p_k Q(x, \xi^k)$ and

$$Q(x, \xi^k) = \begin{cases} \min & {q^k}^\top y \\ \text{s.t.} & W^k y = h^k - T^k x \\ & y \geq 0 \end{cases} = \begin{cases} \max & \pi^\top (h^k - T^k x) \\ \text{s.t.} & {W^k}^\top \pi \leq q^k \end{cases}$$

Intermediate results

(independent of ditribution being continuous or discrete)

- $\partial i_X(\bar{x}) = N_X(\bar{x})$
- $N_X(\bar{x}) = \left\{ \nu \in \mathbb{R}^{nx} : \nu = -A^\top \mu - p \text{ for } \begin{array}{l} \mu \in \mathbb{R}^{nb}, p \in \mathbb{R}^{nx} \\ p \geq 0, p^\top \bar{x} = 0 \end{array} \right\}$
- $f : \mathbb{R}^{nx} \to \mathbb{R} \cup \{+\infty\}$ proper and convex.
  Then $\bar{x}$ minimizes $f \iff 0 \in \partial f(\bar{x})$
- If f_1 and f_2 are polyhedral, then
  $\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x})$
- If dom $f_1 \cap$int(dom $f_2) \neq \emptyset$, then
  $\partial(f_1 + f_2)(\bar{x}) = \partial f_1(\bar{x}) + \partial f_2(\bar{x})$

(necessary and sufficient) Optimality Condition

Consider $\bar{x} \in X$ such that $\phi(\bar{x})$ is finite, $\Omega = \{\omega^1, \ldots, \omega^K\}$

Theorem: $\bar{x}$ solves the 2SLP $\iff \exists \bar{\pi}^k, k = 1, \ldots K$ and $\bar{\mu} \in \mathbb{R}^{nb}$ such that

- $\bar{\pi}^k \in \operatorname{argmax}\{\pi^\top(h^k - T^k\bar{x}) : W^{k\top}\pi \le q^k\}$
- $\displaystyle\sum_{k=1}^{K} p_k T^{k\top}\bar{\pi}^k + A^\top\bar{\mu} \le c$
- $\displaystyle\bar{x}^\top\left(c - \sum_{k=1}^{K} p_k T^{k\top}\bar{\pi}^k - A^\top\bar{\mu}\right) = 0$

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Consider $\bar{x} \in X$ such that $\phi(\bar{x})$ is finite, $\Omega = \{\omega^1, \ldots, \omega^K\}$

Theorem: $\bar{x}$ solves the 2SLP $\Longleftrightarrow$ $\exists \bar{\pi}^k, k = 1, \ldots K$ and $\bar{\mu} \in \mathbb{R}^{nb}$ such that

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Can be shown writing the OC for the large LP

$$\begin{cases} \min & c^\top x + \sum_{k=1}^{K} q^{k\top} y^k \\ \text{s.t.} & Ax = b, x \geq 0 \\ & T^k x + W^k y^k = h^k, k = 1, \ldots, K \\ & y^k \geq 0, k = 1, \ldots, K \end{cases}$$

(necessary and sufficient) Optimality Condition

For continuous distribution, **simple recourse:**

$$\min_{x\in X} c^\top x + \phi(x) \qquad \text{for} \qquad X := \{x \geq 0 : Ax = b\}, \qquad b \in \mathbb{R}^{nb}$$

where $\phi(x) = \mathbb{E}\left[Q(x,\xi)\right] = \int_{-\infty}^{+\infty} Q(x,\xi(\omega))\,d\mathbb{P}(\omega)$ and

$$Q(x,\xi) = \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h(\omega) - Tx \\ & y \geq 0 \end{cases} = \begin{cases} \max & \pi^\top(h(\omega) - Tx) \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

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domϕ =**pos** $W = \mathbb{R}^m$

(necessary and sufficient) Optimality Condition

Consider $\bar{x} \in X$ such that $\phi(\bar{x})$ is finite, with Ω continuous, and recourse simple and fixed (ϕ is proper and $\text{int}(\text{dom}\phi) \cap X \neq \emptyset$).

Theorem: $\bar{x}$ solves the 2SLP $\iff \exists \bar{\pi}(\omega)$, a measurable function, and $\bar{\mu} \in \mathbb{R}^{nb}$ such that

- $\bar{\pi}(\omega) \in \text{argmax}\{\pi^\top (h(\omega) - T\bar{x}) : W^\top \pi \leq q\}$, for $\omega \in \Omega$
- $T^\top \mathbb{E}[\bar{\pi}] + A^\top \bar{\mu} \leq c$
- $\bar{x}^\top \left(c - T^\top \mathbb{E}[\bar{\pi}] - A^\top \bar{\mu} \right) = 0$