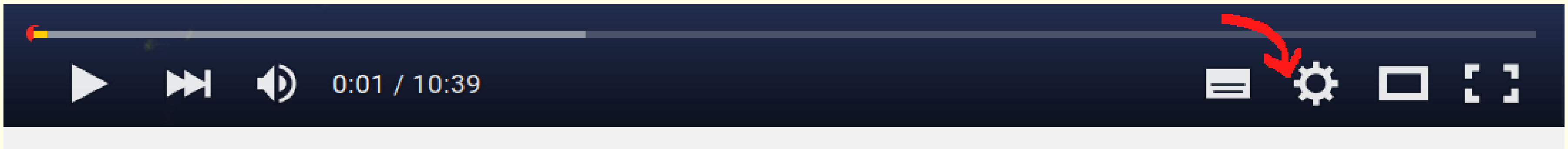
Two-stage Stochastic Linear Programming Problems (two-stage SLP)

Claudia Sagastizábal

BAS Lecture 4, March 17, 2016, IMPA

SVAN 2016

Set YouTube resolution to 480p



for best viewing

The recourse function: LP duality

$$Q(x,\xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} \max & \pi^\top (h - Tx) \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

are dual: same optimal value if both feasible, finite value implies existence of solutions, etc

We shall study the function

$$Q(x,\xi) = \max\left\{\pi^\top (h - Tx) : \pi \in \Pi(q)\right\}$$

where $\Pi(q) := \{\pi : W^\top \pi \leq q\}$

Q **is the maximum of affine functions of** x**,**

maximum taken over a polyhedral set

A Glimpse of Convex Analysis

- **SETS**
  - Recession cone
  - Polar cone
- **FUNCTIONS**
  - Indicator function
  - Support function
  - Conjugate function
  - Piecewise and polyhedral functions

– Subdifferential

The recourse function: composition

$$Q(x,\xi) = \begin{cases} \inf & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} \sup & \pi^\top (h - Tx) \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

$$\begin{aligned} \mathbb{R}^m \ni z \mapsto v(z) : &= \inf\left\{ q^\top y : Wy = z \text{ for } y \geq 0 \right\} \\ &= \inf\left\{ q^\top y : z \in \text{pos}(W) \right\} \end{aligned}$$

where $\text{pos}(W) := \{z \in \mathbb{R}^m : z = Wy \text{ for some } y \geq 0\}$

$\Longrightarrow$

$$Q(x,\xi) = v(h - Tx)$$

The function v: domain

$v(z) = \inf\{q^\top y : Wy = z \text{ for } y \geq 0\}$ and $\begin{cases} \sup & \pi^\top z \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$

If $z \in \text{pos}(W)$ the primal feasible set is not empty. If the dual feasible is not empty too, then $v(z)$ is finite and

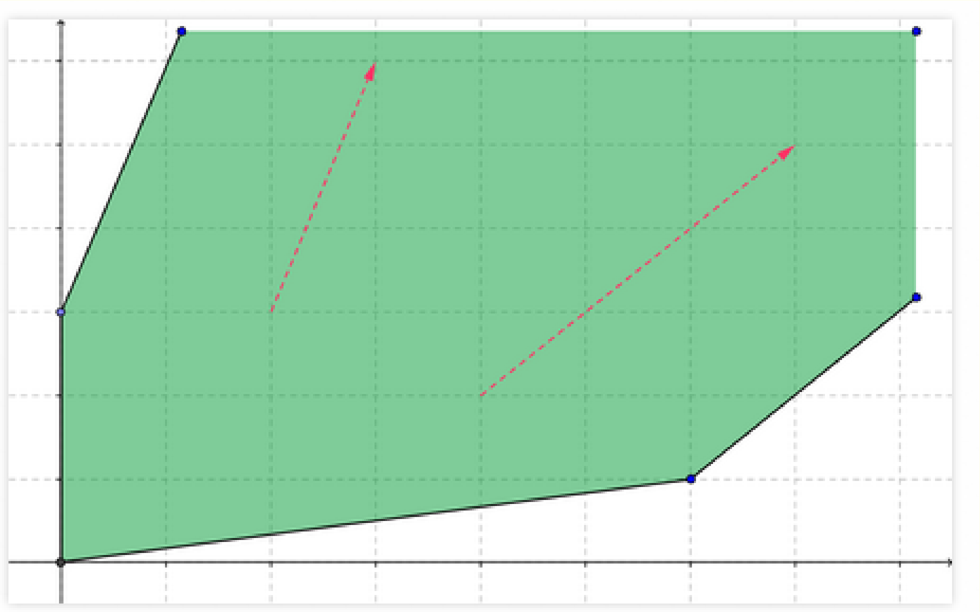
$$v(z) = \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = z \\ & y \geq 0 \end{cases} = \begin{cases} \max & \pi^\top z \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

$$= \begin{cases} \max & \pi^\top z \\ \text{s.t.} & \pi \in \Pi \end{cases}$$

where $\Pi = \Pi(q) = \{\pi \in \mathbb{R}^m : W^\top \pi \leq q\}$ is a polyhedron

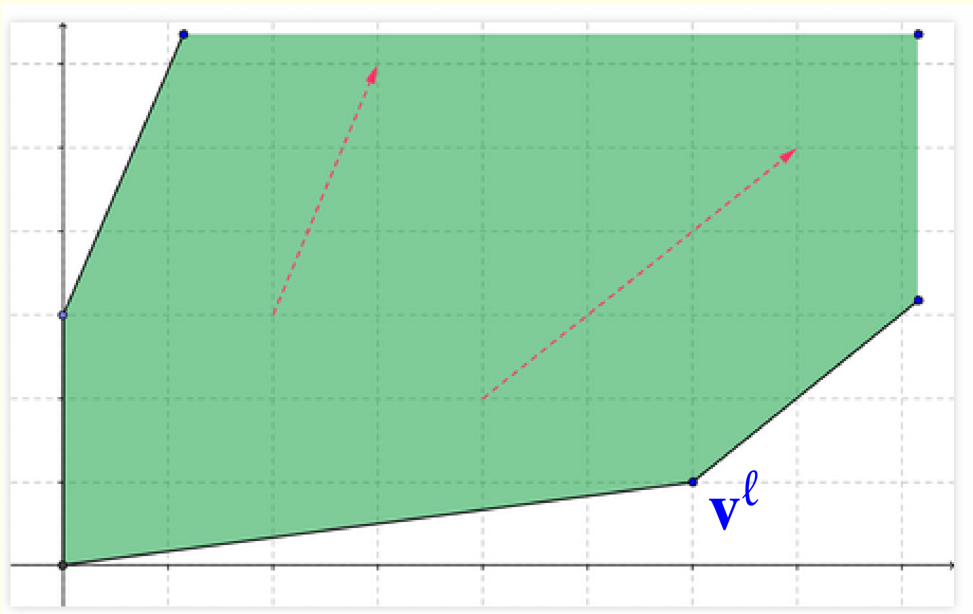
The polyhedron Π

$$\begin{aligned} \Pi &= \{\pi \in \mathbb{R}^m : W^\top \pi \le q\} \\ &= \mathrm{conv}\{v^1, \ldots, v^L\} + \Pi_\infty \end{aligned}$$



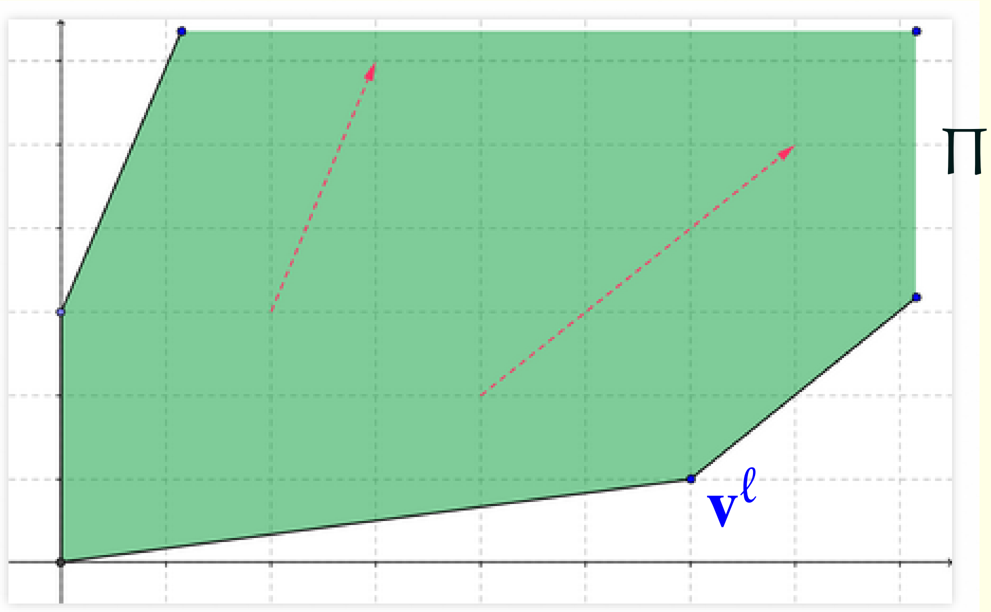
The polyhedron Π

$$\begin{aligned}\Pi &= \{\pi \in \mathbb{R}^m : W^\top \pi \leq q\} \\ &= \mathrm{conv}\{v^1, \ldots, v^L\} + \Pi_\infty\end{aligned}$$



The polyhedron Π

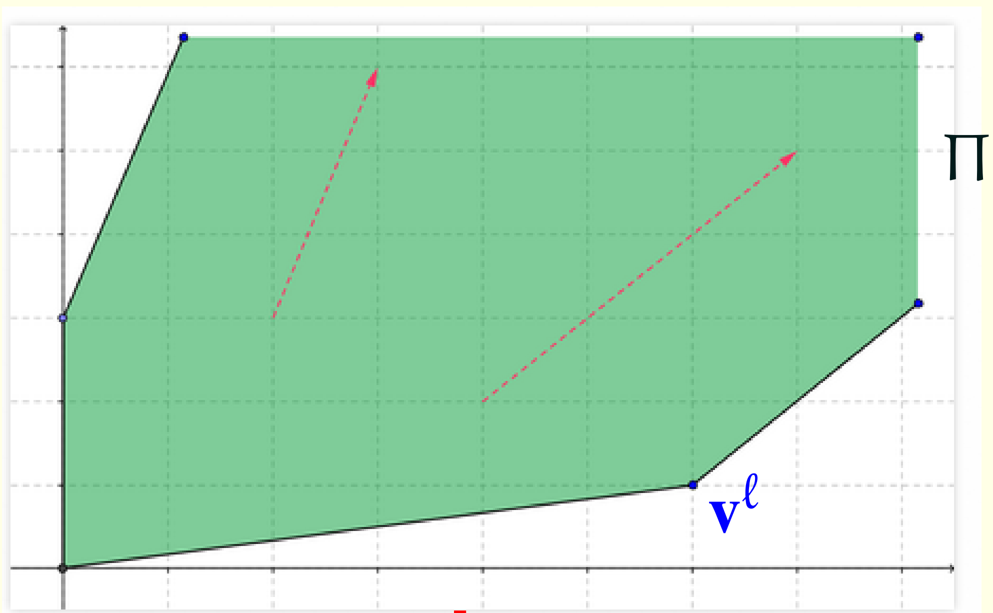
$$
\begin{aligned}
\Pi &= \{\pi \in \mathbb{R}^m : W^\top \pi \leq q\} \\
&= \mathrm{conv}\{v^1, \ldots, v^L\} + \Pi_\infty
\end{aligned}
$$



$$\Pi_\infty = \{d \in \mathbb{R}^m : W^\top d \leq 0\}$$

The polyhedron Π

$$\begin{aligned}\Pi &= \{\pi \in \mathbb{R}^m : W^\top \pi \leq q\} \\ &= \mathrm{conv}\{v^1, \ldots, v^L\} + \Pi_\infty\end{aligned}$$

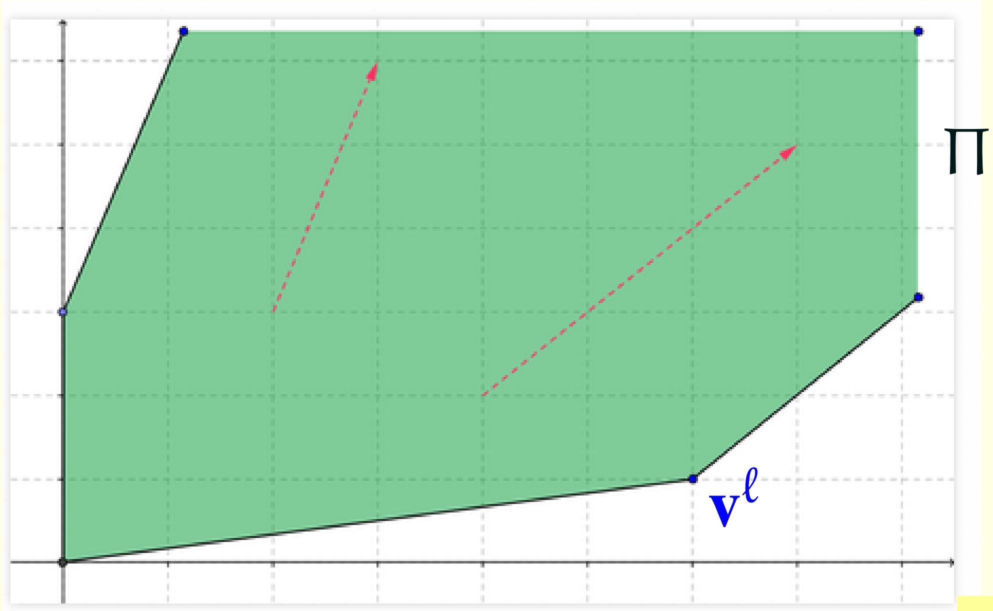


$$\Pi_\infty = \{d \in \mathbb{R}^m : W^\top d \leq 0\}$$

Π is a polytope **+** a recession cone

The polyhedron Π

$$\begin{aligned}\Pi &= \{\pi \in \mathbb{R}^m : W^\top \pi \le q\} \\ &= \mathrm{conv}\{v^1, \ldots, v^L\} + \Pi_\infty\end{aligned}$$



$$\Pi_\infty = \{d \in \mathbb{R}^m : W^\top d \le 0\}$$

Π is a polytope + a recession cone

$$\pi = \sum_{\ell=1}^{L} \alpha_\ell v^\ell + d$$

$$\alpha \in \Delta^{L} := \{\alpha_\ell \geq 0, \sum_{\ell=1}^{L} \alpha_\ell = 1\}$$

The subdifferential of $v(z) = \left\{ \begin{array}{ll} \max & \pi^\top z \\ \textbf{s.t.} & \pi = \sum_{\ell=1}^{L} \alpha_\ell v^\ell + \mathrm{d} \end{array} \right.$

$$v(z) \quad = \quad \left\{ \begin{array}{ll} \max & \sum_{\ell=1}^{L} \alpha_\ell v^{\ell\top} z \\ \text{s.t.} & \alpha \in \Delta^L \end{array} \right. \quad + \quad \left\{ \begin{array}{ll} \max & \mathrm{d}^\top z \\ \text{s.t.} & W^\top \mathrm{d} \leq 0 \end{array} \right.$$

The subdifferential of $v(z) = \begin{cases} \max & \pi^\top z \\ \textbf{s.t.} & \pi = \sum_{\ell=1}^{L} \alpha_\ell v^\ell + d \end{cases}$

$$v(z) = \begin{cases} \max & \sum_{\ell=1}^{L} \alpha_\ell v^{\ell\top} z \\ \text{s.t.} & \alpha \in \Delta^L \end{cases} + \begin{cases} \max & d^\top z \\ \text{s.t.} & W^\top d \le 0 \end{cases}$$

$$= \begin{cases} \max & v^{\ell\top} z \\ \text{s.t.} & \ell = 1, \ldots, L \end{cases} + \begin{cases} 0 & \text{if } d^\top z \le 0 \\ +\infty & \text{if } d^\top z > 0 \end{cases}$$

The subdifferential of $v(z) = \left\{ \begin{array}{ll} \max & \pi^\top z \\ \textbf{s.t.} & \pi = \sum_{\ell=1}^{L} \alpha_\ell v^\ell + d \end{array} \right.$

$$v(z) = \left\{ \begin{array}{ll} \max & \sum_{\ell=1}^{L} \alpha_\ell v^{\ell\top} z \\ \text{s.t.} & \alpha \in \Delta^L \end{array} \right. + \left\{ \begin{array}{ll} \max & d^\top z \\ \text{s.t.} & W^\top d \le 0 \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \max & v^{\ell\top} z \\ \text{s.t.} & \ell = 1, \ldots, L \end{array} \right. + \left\{ \begin{array}{ll} 0 & \text{if } d^\top z \le 0 \\ +\infty & \text{if } d^\top z > 0 \end{array} \right.$$

= a piecewise linear function + an indicator function

The subdifferential of $v(z) = \begin{cases} \max & \pi^\top z \\ \textbf{s.t.} & \pi = \sum_{\ell=1}^{L} \alpha_\ell v^\ell + \mathrm{d} \end{cases}$

$$v(z) = \begin{cases} \max & \sum_{\ell=1}^{L} \alpha_\ell v^{\ell\top} z \\ \text{s.t.} & \alpha \in \Delta^L \end{cases} + \begin{cases} \max & \mathrm{d}^\top z \\ \text{s.t.} & W^\top \mathrm{d} \le 0 \end{cases}$$

$$= \begin{cases} \max & v^{\ell\top} z \\ \text{s.t.} & \ell = 1, \ldots, L \end{cases} + \begin{cases} 0 & \text{if } \mathrm{d}^\top z \le 0 \\ +\infty & \text{if } \mathrm{d}^\top z > 0 \end{cases}$$

= a piecewise linear function + an indicator function

$$\partial \mathbf{v}(\mathbf{z_0}) = \mathbf{conv}\left\{ \mathbf{v}^\ell : \mathbf{v}^{\ell\top} \mathbf{z_0} = \mathbf{v}(\mathbf{z_0}) \right\}$$

The subdifferential of $v(z) = \left\{ \begin{array}{ll} \max & \pi^\top z \\ \textbf{s.t.} & \pi = \sum_{\ell=1}^{L} \alpha_\ell v^\ell + d \end{array} \right.$

$$v(z) = \left\{ \begin{array}{ll} \max & \sum_{\ell=1}^{L} \alpha_\ell v^{\ell\top} z \\ \text{s.t.} & \alpha \in \Delta^L \end{array} \right. + \left\{ \begin{array}{ll} \max & d^\top z \\ \text{s.t.} & W^\top d \le 0 \end{array} \right.$$

$$= \left\{ \begin{array}{ll} \max & v^{\ell\top} z \\ \text{s.t.} & \ell = 1, \ldots, L \end{array} \right. + \left\{ \begin{array}{ll} 0 & \text{if } d^\top z \le 0 \\ +\infty & \text{if } d^\top z > 0 \end{array} \right.$$

= a piecewise linear function + an indicator function

$$\partial \mathbf{v}(\mathbf{z_0}) = \mathbf{conv}\left\{ \mathbf{v}^\ell : \mathbf{v}^{\ell\top} \mathbf{z_0} = \mathbf{v}(\mathbf{z_0}) \right\} = \arg\max \left\{ \pi^\top \mathbf{z_0} : \pi \in \Pi \right\}$$

The subdifferential of $Q(\cdot, \xi)$

Since $Q(x_0, \xi) = v(h - Tx_0)$, a chain rule gives

$$\partial \mathbf{Q}(\mathbf{x_0}, \xi) = -\mathbf{T}^\top \arg\max\left\{\pi^\top(\mathbf{h} - \mathbf{T}\mathbf{x_0}) : \pi \in \Pi\right\}$$

The subdifferential of $Q(\cdot, \xi)$

Since $Q(x_0, \xi) = v(h - Tx_0)$, a chain rule gives

$$\partial \mathbf{Q}(\mathbf{x_0}, \xi) = -\mathbf{T}^\top \arg\max \left\{ \pi^\top (\mathbf{h} - \mathbf{T}\mathbf{x_0}) : \pi \in \Pi \right\}$$

The subdifferential of $\mathbb{E}[Q(\cdot, \xi)]$

The subdifferential of $Q(\cdot, \xi)$

Since $Q(x_0, \xi) = \nu(h - Tx_0)$, a chain rule gives

$$\partial \mathbf{Q}(\mathbf{x_0}, \xi) = -\mathbf{T}^\top \arg\max \left\{ \pi^\top (\mathbf{h} - \mathbf{T}\mathbf{x_0}) : \pi \in \Pi \right\}$$

The subdifferential of $\mathbb{E}[Q(\cdot, \xi)]$

- ξ has a finite support: K scenarios

- ξ is a general distribution

The subdifferential of $Q(\cdot, \xi)$

Since $Q(x_0, \xi) = v(h - Tx_0)$, a chain rule gives

$$\partial \mathbf{Q}(\mathbf{x_0}, \xi) = -\mathbf{T}^\top \arg\max \left\{ \pi^\top (\mathbf{h} - \mathbf{T}\mathbf{x_0}) : \pi \in \Pi \right\}$$

The subdifferential of $\mathbb{E}[Q(\cdot, \xi)]$

- ξ has a finite support: K scenarios

  easier case

- ξ is a general distribution

  less easy case: depends on recourse structure

The subdifferential of $\mathbb{E}[Q(\cdot, \xi)]$

ξ has a finite support: K scenarios

$$\partial \mathbb{E}[\mathbf{Q}(\mathbf{x_0}, \xi)] = -\sum_{\mathbf{k=1}}^{\mathbf{K}} \mathbf{p_k} \mathbf{T^k}^{\top} \arg\max \left\{ \pi^{\top} (\mathbf{h^k} - \mathbf{T^k} \mathbf{x_0}) : \pi \in \Pi(\mathbf{q^k}) \right\}$$

Homework

For the oil production example, considering RHS uncertainty only of the form

- $h_1 = 180 + \zeta_1$ com $\zeta_1 \sim \mathcal{N}(0, 16)$.
- $h_2 = 163 + \zeta_2$ com $\zeta_2 \sim \mathcal{N}(0, 9)$.

use Matlab/Octave to generate K demand scenarios and solve the models

1. Deterministic
2. Worst-case
3. Scenario Analysis
4. Chance-constrained

5. With recourse, considering that the gasoline bought in case of shortage costs 7 and 12.

Try your models first with $K = 1$ and then run the code with $K = 10$ and $K = 100$ scenarios.

Compare the different optimal values and solutions found.

To assess the quality solution, compute the reliability of each $\bar{x}$, using the Matlab function `mvncdf`.

Questions on the homework (mandatory for PhD students)

- Worst-Case scenario: is it ok to take an interval for 99% confidence and then add the maximum value on that interval to the oil demand?

- Scenario Analysis: what would the final answer be here?

Questions on the homework (mandatory for PhD students)

- Worst-Case scenario: is it ok to take an interval for 99% confidence and then add the maximum value on that interval to the oil demand?

  Yes

- Scenario Analysis: what would the final answer be here?

  We just solve K times the problem for K different scenarios.

  One could build a sort of “worst-case solution",
  based on the K scenarios