TWO-STAGE STOCHASTIC LINEAR PROGRAMMING PROBLEMS (TWO-STAGE SLP)

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BAS Lecture 3, March 15, 2016, IMPA

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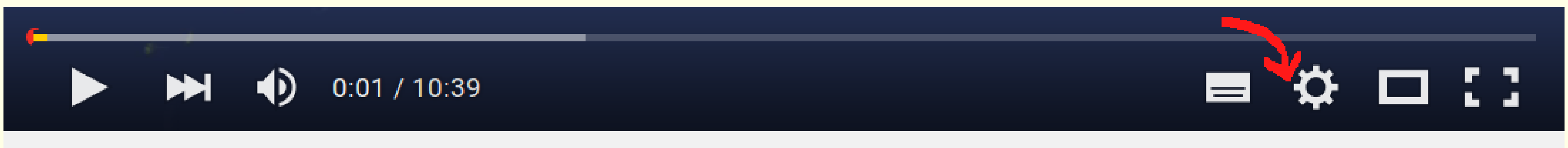
Set YouTube resolution to 480p



for best viewing

Questions on the homework (mandatory for PhD students)

- Worst-Case scenario: is it ok to take an interval for 99% confidence and then add the maximum value on that interval to the oil demand?

- Scenario Analysis: what would the final answer be here? We just solve K times the problem for K different scenarios.

Two-stage SLP

The recourse model for our oil production problem

$$
\left\{
\begin{array}{lll}
\min & 2x_1 + 3x_2 + \sum_{k=1}^{K} p_k \left[7y_1(\omega_k) + 12y_2(\omega_k)\right] & \\
\text{s.t.} & x_1 + x_2 \le 100, \quad x_1, x_2 \ge 0, & \\
 & \alpha(\omega_k)x_1 + 6x_2 + y_1(\omega_k) = h_1(\omega_k), & k = 1, \ldots, K \\
 & 3x_1 + \beta(\omega_k)x_2 + y_2(\omega_k) = h_2(\omega_k), & k = 1, \ldots, K \\
 & y_1(\omega_k), y_2(\omega_k) \ge 0, & k = 1, \ldots, K.
\end{array}
\right.
$$

Two-stage SLP

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Two-stage SLP

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Two-stage SLP

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For **W** the **recourse** matrix

Two-stage SLP: what's uncertain?

The recourse model for our oil production problem

$$\begin{cases} \min & c^\top x + \sum_{k=1}^{K} p_k \left[7y_1(\omega_k) + 12y_2(\omega_k)\right] \\ \text{s.t.} & Ax = b, \quad x \geq 0, \\ & \mathbf{T}(\omega_\mathbf{k})x + Wy(\omega_k) = \mathbf{h}(\omega_\mathbf{k}), \qquad k = 1, \ldots, K \\ & y(\omega_k) \geq 0, \qquad k = 1, \ldots, K. \end{cases}$$

has uncertain **T** and **h**

Two-stage SLP: what's uncertain?

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$$\left\{ \begin{array}{lll} \min & c^\top x + \sum_{k=1}^{K} p_k \left[\mathbf{7} y_1(\omega_k) + \mathbf{12} y_2(\omega_k) \right] & \\ \text{s.t.} & Ax = b, \quad x \geq 0, & \\ & \mathbf{T}(\omega_\mathbf{k}) x + W y(\omega_k) = \mathbf{h}(\omega_\mathbf{k}), & k = 1, \ldots, K \\ & y(\omega_k) \geq 0, & k = 1, \ldots, K. \end{array} \right.$$

has uncertain **T** and **h**

the wait-and-see cost can be uncertain too: **q**

Two-stage SLP: what's uncertain?

The recourse model for our oil production problem

$$\begin{cases} \min & c^\top x + \sum_{k=1}^{K} p_k \mathbf{q}(\omega_{\mathbf{k}})^\top y(\omega_k) \\ \text{s.t.} & Ax = b, \quad x \geq 0, \\ & \mathbf{T}(\omega_{\mathbf{k}})x + Wy(\omega_k) = \mathbf{h}(\omega_{\mathbf{k}}), \quad k = 1, \ldots, K \\ & y(\omega_k) \geq 0, \qquad\qquad k = 1, \ldots, K. \end{cases}$$

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likewise for the recourse matrix $\mathbf{W}$.

Two-stage SLP: simplified notation

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drop all $\omega_\mathbf{k}$

has uncertain **T** and **h**

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drop all ω_k

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\begin{cases}
\min \quad c^\top x + \sum_{k=1}^{K} p_k \mathbf{q}(\omega_k)^\top y \\
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\qquad \mathbf{T}(\omega_k) x + \mathbf{W}(\omega_k) y(\omega_k) = \mathbf{h}(\omega_k), \quad k = 1, \ldots, K \\
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consider ω

in a probability space

has uncertain $\mathbf{T}$ and $\mathbf{h}$

the wait-and-see cost can be uncertain too: $\mathbf{q}$

likewise for the recourse matrix $\mathbf{W}$.

Two-stage SLP: simplified notation

The recourse model for our oil production problem

consider ω

in a probability space

$$\begin{cases} \min & c^\top x + \mathbb{E}\left[\, \mathbf{q}(\omega_k)^\top y \,\right] \\ \text{s.t.} & Ax = b, \quad x \geq 0, \\ & \mathbf{T}(\omega_k)x + \mathbf{W}(\omega_k)y = \mathbf{h}(\omega_k) \quad \text{a.e.}, \\ & y \geq 0, \quad \text{a.e.} \end{cases}$$

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define $\xi = (q, T, W, h)$ **a random vector** $\xi = \xi(\omega)$

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define $\xi = (q, T, W, h)$ **a random vector** $\xi = \xi(\omega)$

and REMEMBER $y = y(\omega)$ **is a random variable in this setting!**

Two-stage SLP

$$\begin{cases} \min & c^\top x + \mathbb{E}\left[q^\top y\right] \\ \text{s.t.} & Ax = b\,, \quad x \geq 0\,, \\ & Tx + Wy = h \qquad\qquad \text{a.e.,} \\ & y \geq 0 \qquad\qquad\qquad\quad \text{a.e.} \end{cases}$$

y lives in a prob.space

Two-stage SLP: reformulation

$$
\begin{cases}
\min & c^\top x + \mathbb{E}\left[q^\top y \ \textcolor{red}{\mathbf{xxxx}} \ Q(x, \xi) \right] \\
\text{s.t.} & Ax = b\,, \quad x \geq 0\,,
\end{cases}
$$

Two-stage SLP: reformulation

$$\begin{cases} \min & c^\top x + \mathbb{E}\left[\cancel{q^\top y}\ Q(x,\xi)\right] \\ \text{s.t.} & Ax = b\,, \quad x \geq 0\,, \end{cases}$$

for the recourse function

$$Q(\mathbf{x},\xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - T\mathbf{x} \\ & y \geq 0 \end{cases}$$

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$$Q(\mathbf{x}, \xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - T\mathbf{x} \\ & y \geq 0 \end{cases}$$

Now ξ is fixed, and y is a VECTOR in $\mathbb{R}^m$!

Two-stage SLP: names

$$\begin{cases} \min & c^\top x + \mathbb{E}\left[Q(x,\xi)\right] \\ \text{s.t.} & Ax = b\,, \quad x \geq 0\,, \end{cases}$$

is the first-stage problem, with first-stage variable x, while for each pair (x,ξ),

$$Q(x,\xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{cases}$$

is a second-stage problem, with second-stage variable y, a simple LP

Two-stage SLP: names

$$\begin{cases} \min & c^\top x + \mathbb{E}\left[Q(x,\xi)\right] \\ \text{s.t.} & Ax = b, \quad x \geq 0, \end{cases}$$

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Two-stage SLP: 1st stage pbm

$$\begin{cases} \min & c^\top x + \mathbb{E}\left[Q(x,\xi)\right] \\ \textbf{s.t.} & Ax = b\,, \quad x \geq 0\,, \end{cases}$$

is the first-stage problem, with first-stage variable x, while for each pair (x,ξ),

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is a second-stage problem, with second-stage variable y, a simple LP **for each pair** (x,ξ)

We need to study the properties of the recourse function

The recourse function

$$Q(x,\xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{cases} = \begin{cases} \max & \pi^\top (h - Tx) \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

– For fixed $\tilde{\xi}$, when is $Q(\cdot, \tilde{\xi})$ finite?

– What does $Q(x, \tilde{\xi}) = -\infty$ mean?

– What does $Q(x, \tilde{\xi}) = +\infty$ mean?

The recourse function

$$Q(x,\tilde{\xi}) := \begin{cases} \min & \tilde{q}^\top y \\ \text{s.t.} & \tilde{W}y = \tilde{h} - \tilde{T}x \\ & y \geq 0 \end{cases}$$

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recall that fixing $\tilde{\xi}$ amounts to taking $\tilde{q}, \tilde{T}, \tilde{W}, \tilde{h}$

The recourse function

$$Q(x,\xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{cases} = \begin{cases} \max & \pi^\top (h - Tx) \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

– For fixed ξ, when is $Q(\cdot,\xi)$ finite?

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recall that fixing ξ amounts to fixing q, T, W, h

The recourse function: LP duality

$$Q(x,\xi) := \begin{cases} \min & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{cases} \qquad \begin{cases} \max & \pi^\top (h - Tx) \\ \text{s.t.} & W^\top \pi \leq q \end{cases}$$

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are dual.

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are dual: same optimal value if both feasible, finite value implies existence of solutions, etc

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We shall study the function

$$Q(x,\xi) = \max\left\{\pi^\top (h - Tx) : \pi \in \Pi(q)\right\}$$

where $\Pi(q) := \{\pi : W^\top \pi \leq q\}$

The recourse function: LP duality

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where $\Pi(q) := \{\pi : W^\top \pi \leq q\}$

Q **is the maximum of affine functions of** x**,**
maximum taken over a polyhedral set

A Glimpse of Convex Analysis

- **SETS**
  - Recession cone
  - Polar cone
- **FUNCTIONS**
  - Indicator function
  - Support function
  - Conjugate function
  - Piecewise and polyhedral functions
  - Subdifferential