ALGORITHMS FOR TWO-STAGE SP: A PRIMER ON NONSMOOTH OPTIMIZATION (SUITE)

Claudia Sagastizábal

BAS Lecture 10, April 12, 2016, IMPA

SVAN 2016

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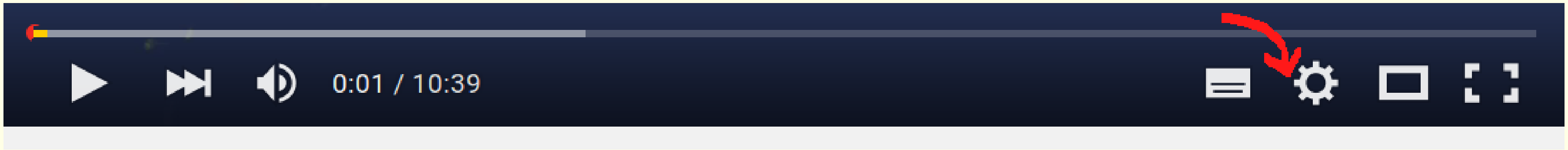

for best viewing

Cutting-plane methods for $\min_{x \in X} f(x)$, X compact polyhedron

To minimize f (unavailable in an explicit manner), minimize its model $\mathbf{M}(x) = \max_i \left\{ f^i + g^{i\top}(x - x^i) \right\}$

Improve the model at each iteration

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$$\begin{aligned} \mathbf{M}_{k+1}(x) &= \max_{i \leq k+1} \left\{ f^i + g^{i\top}(x - x^i) \right\} \\ &= \max\left(\mathbf{M}_k(x), f^{k+1} + g^{k+1\top}(x - x^{k+1}) \right) \\ &\quad \textbf{where } x^{k+1} \textbf{ minimizes } \mathbf{M}_k \end{aligned}$$

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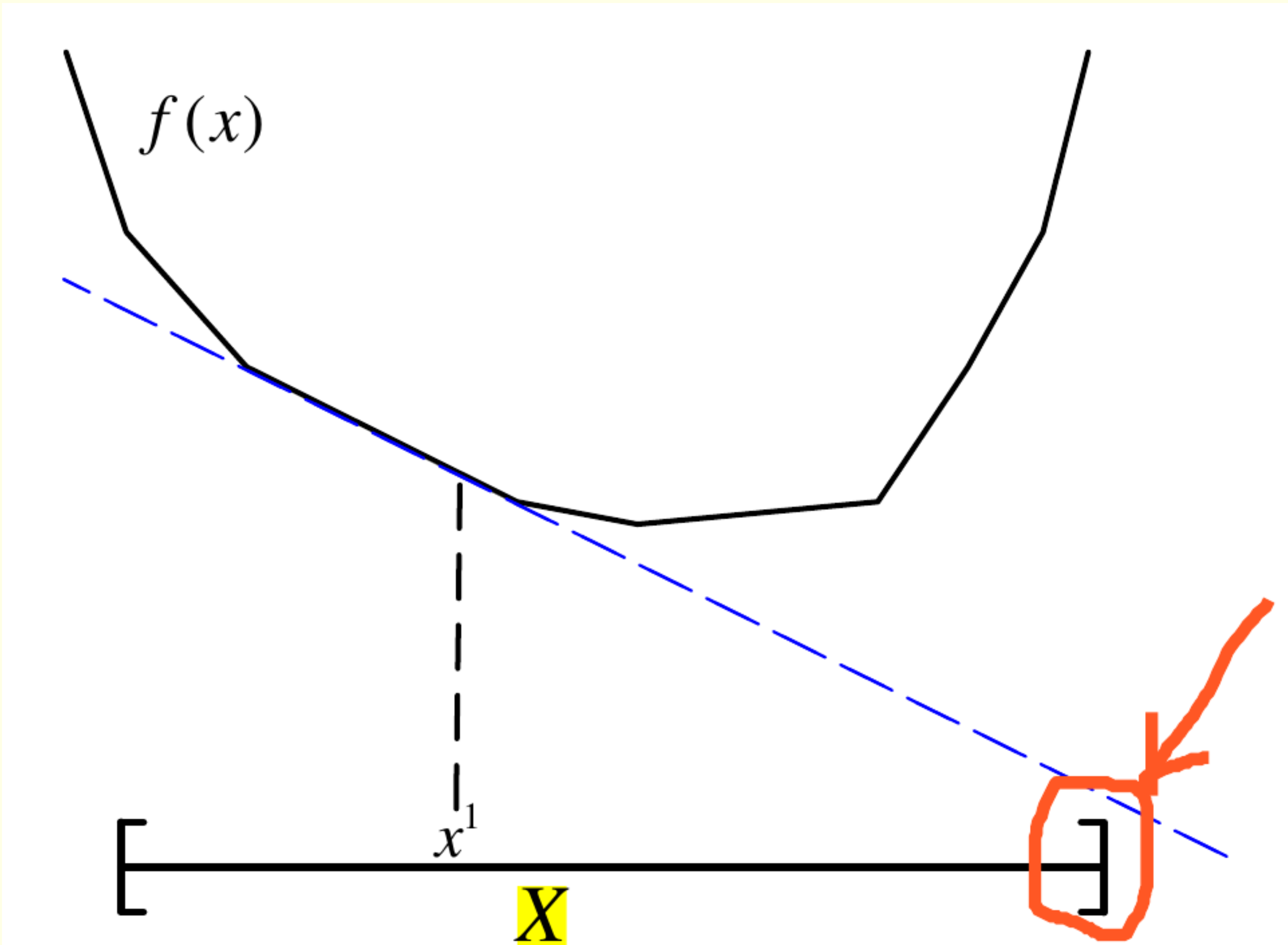
$$

where x^{k+1} minimizes $\mathbf{M}_k$

Instead of $x^* \in \arg\min_X f(x)$ at one shot,

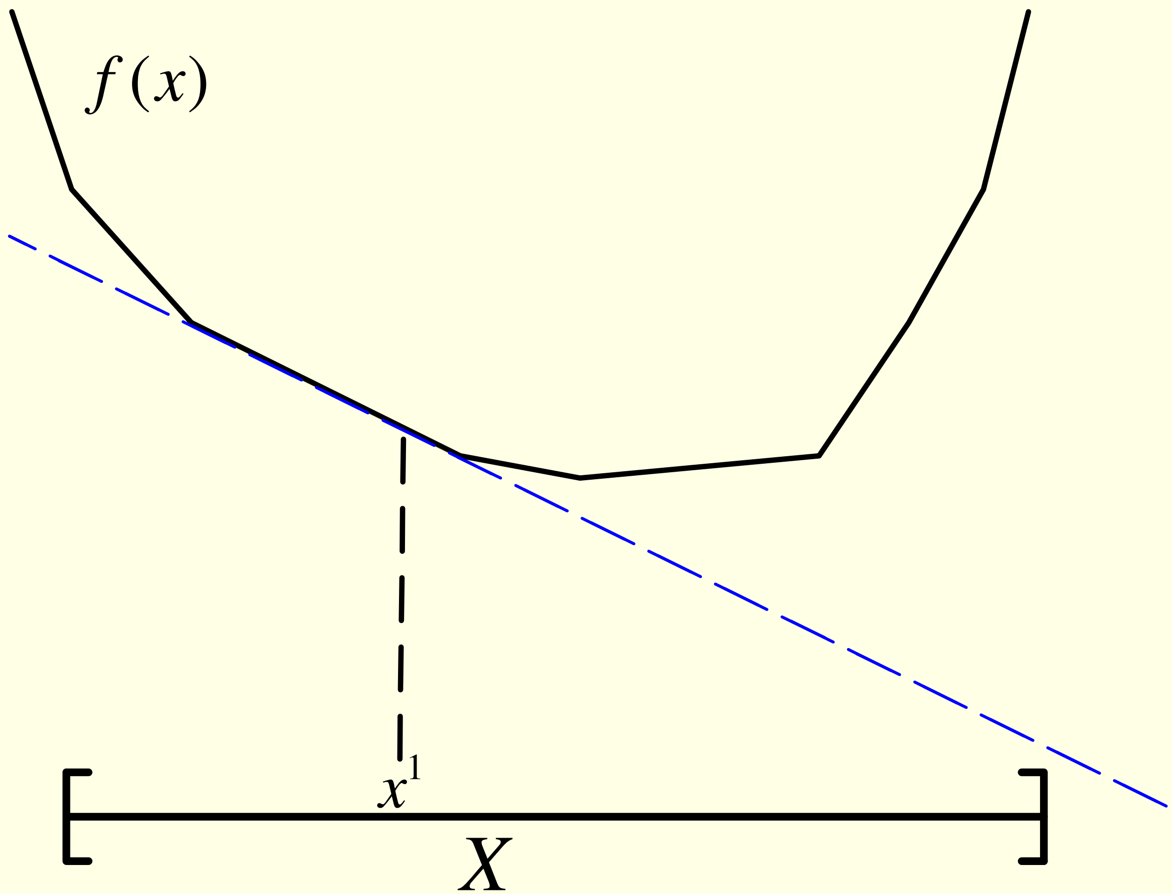
$x^{k+1} \in \arg\min_X \mathbf{M}_k(x)$ **iteratively**

Cutting-plane methods for $\min_{x \in X} f(x)$, X compact polyhedron



may require artificial bounding if X not compact

Cutting-plane methods for $\min_{x \in X} f(x)$, X compact polyhedron



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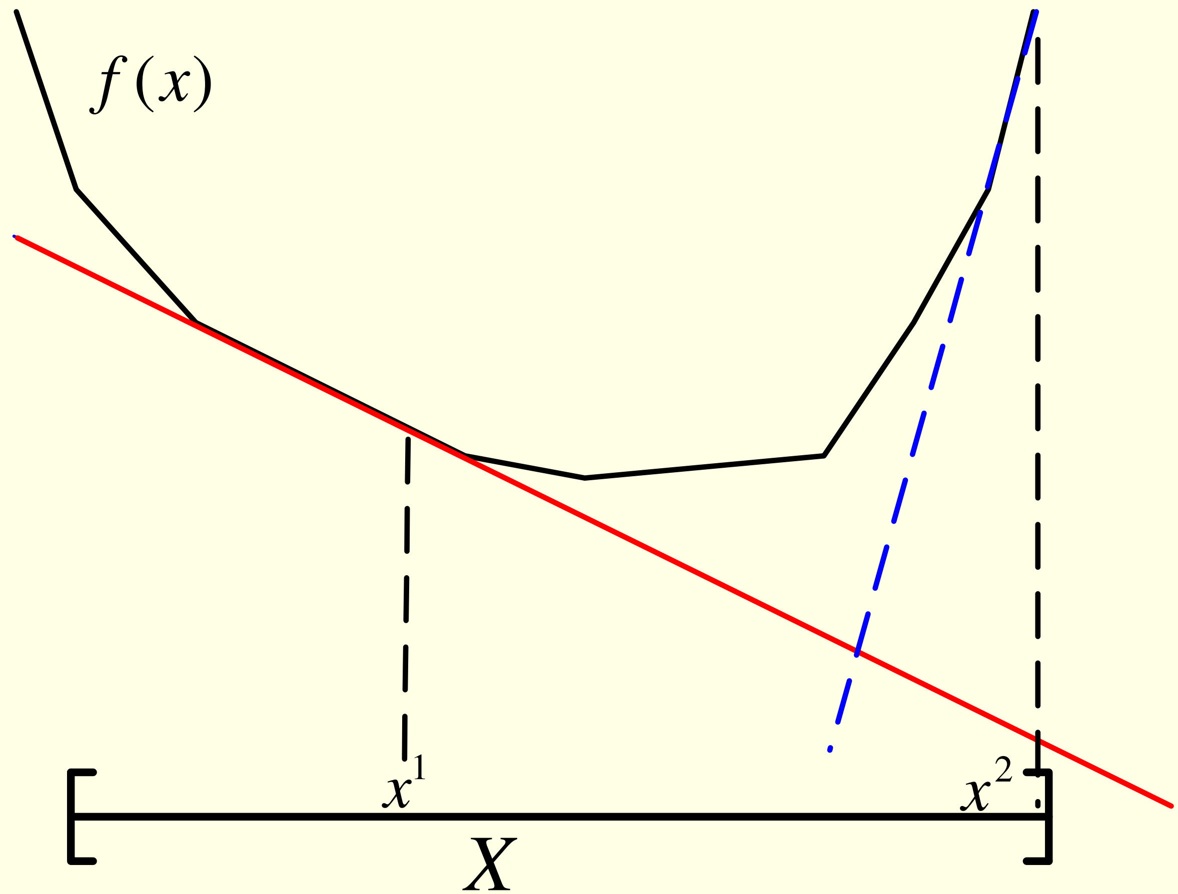

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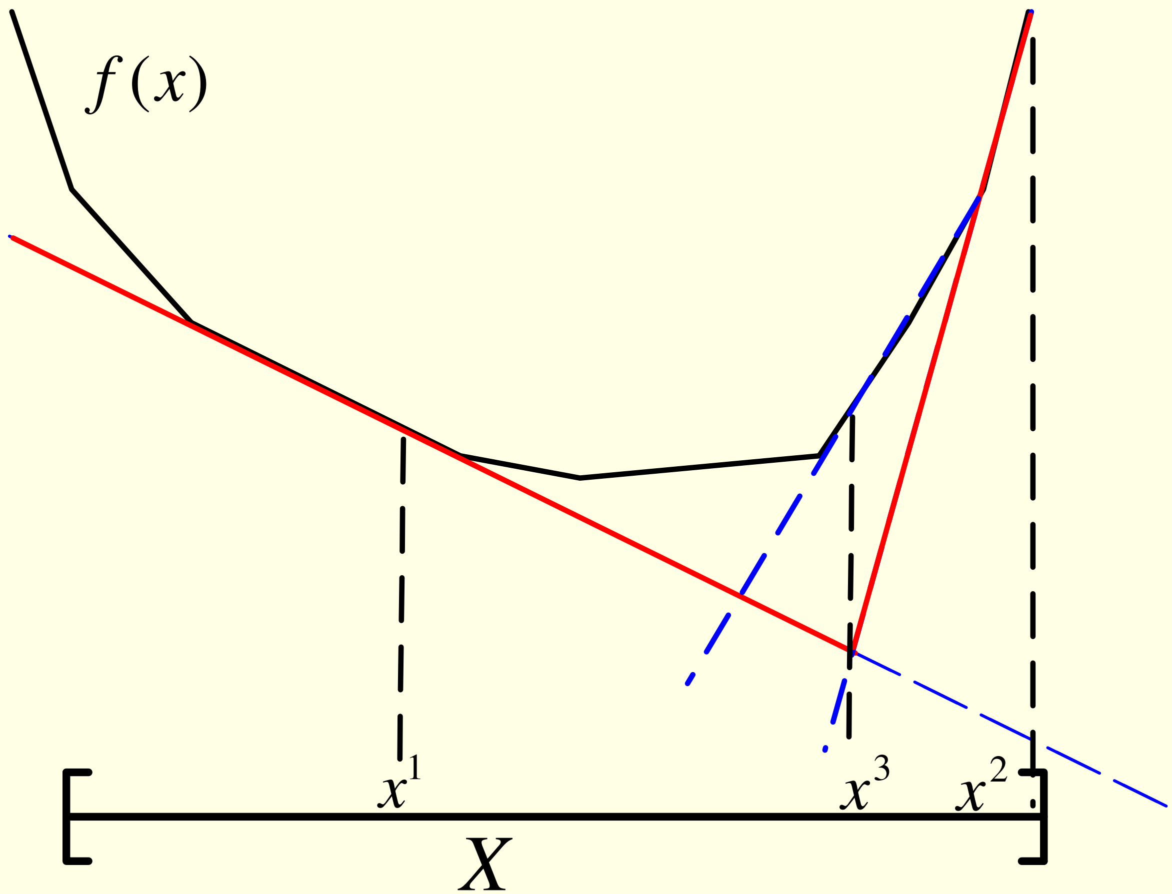

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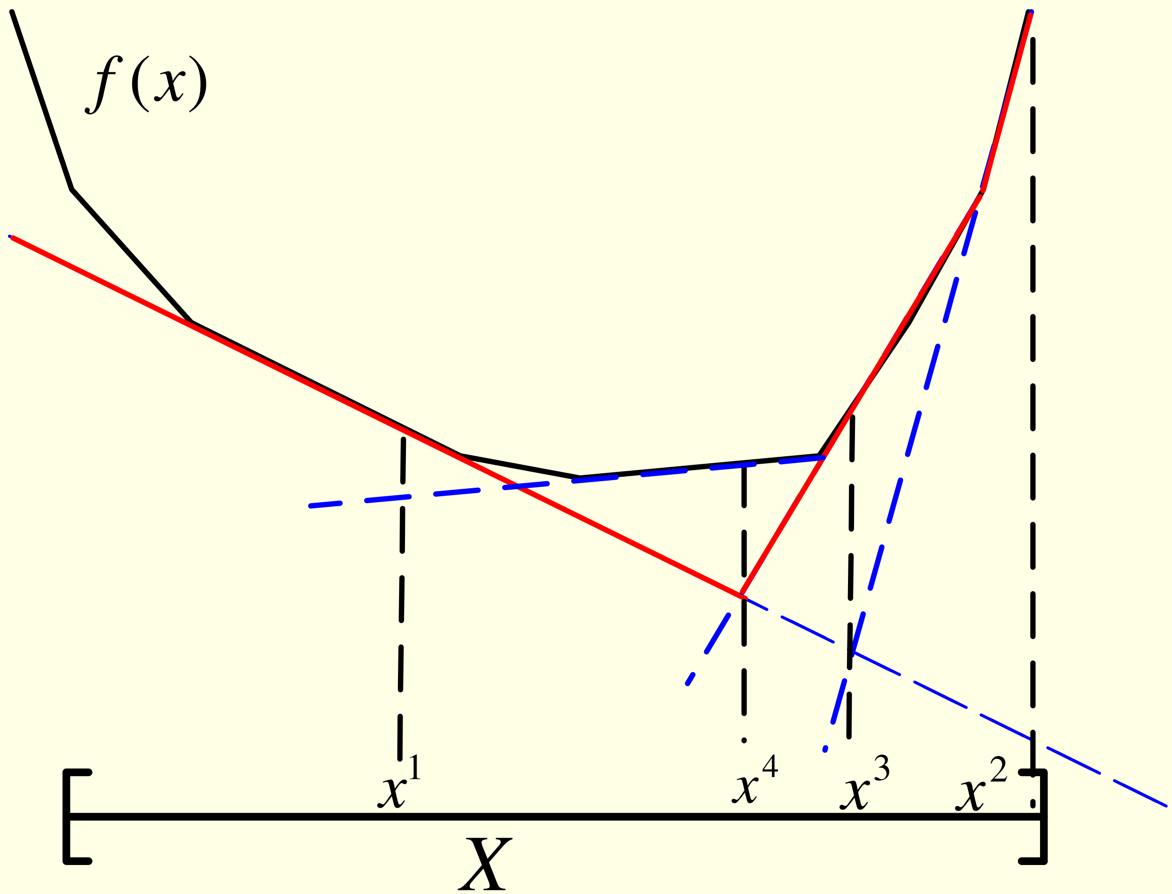

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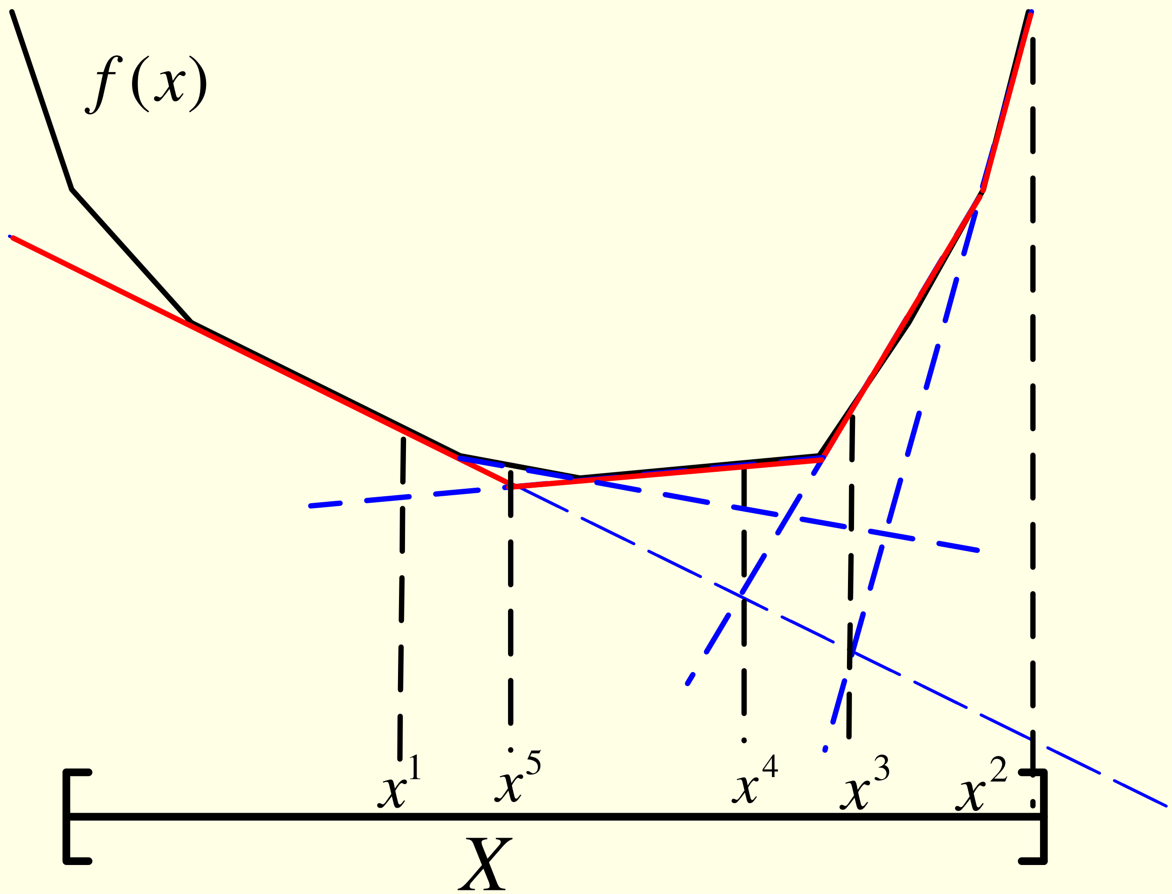

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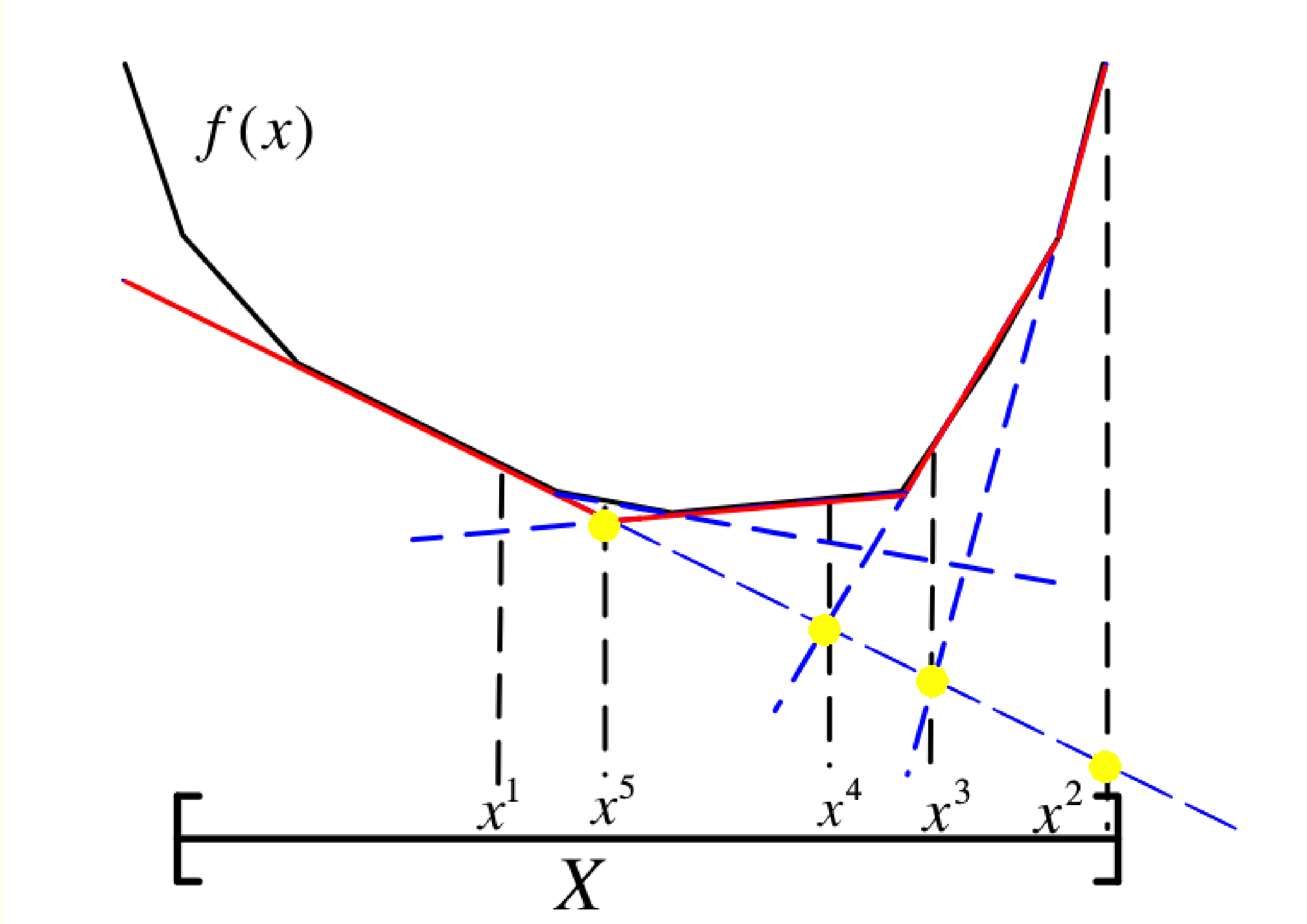

$\{\mathbf{M}_k(x^{k+1})\}$ increases

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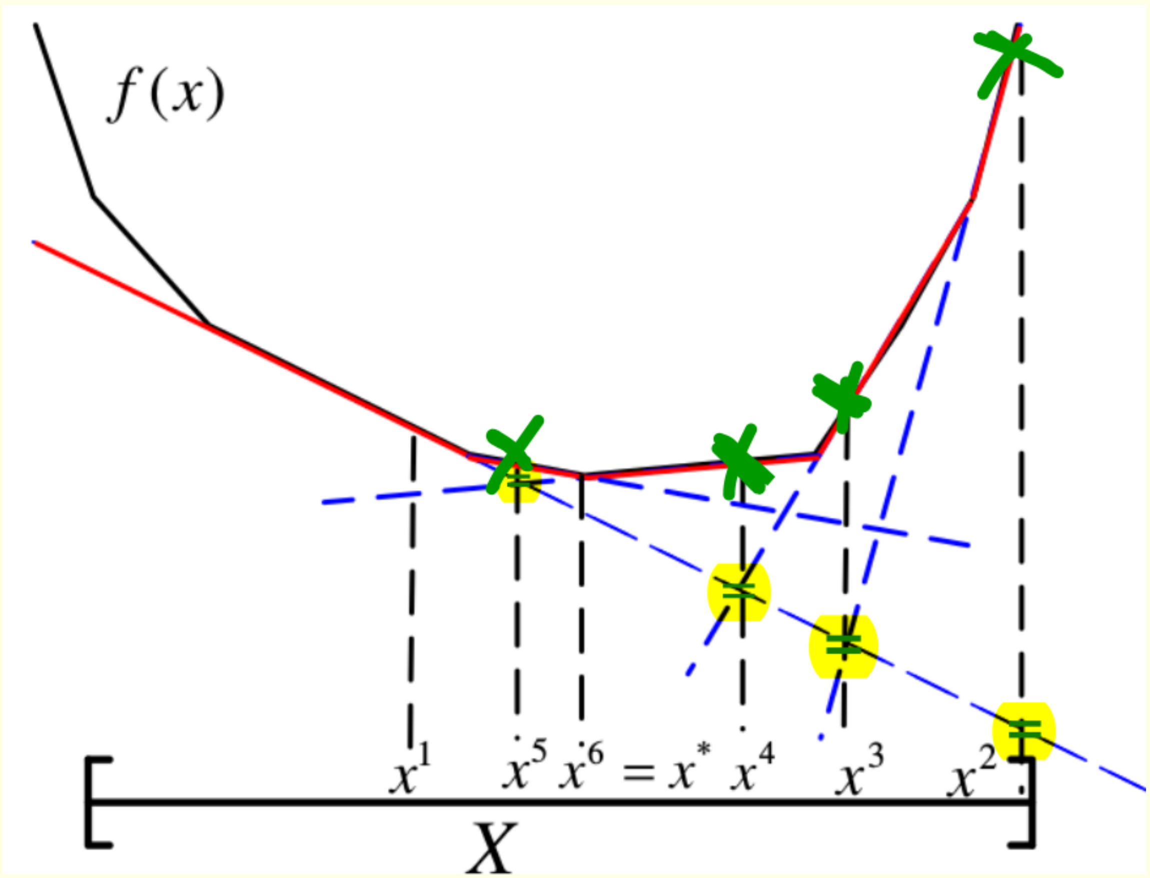

$\{\mathbf{M}_k(x^{k+1})\}$ increases but not necessarily the functional values:

$f(x^5) > f(x^4)$

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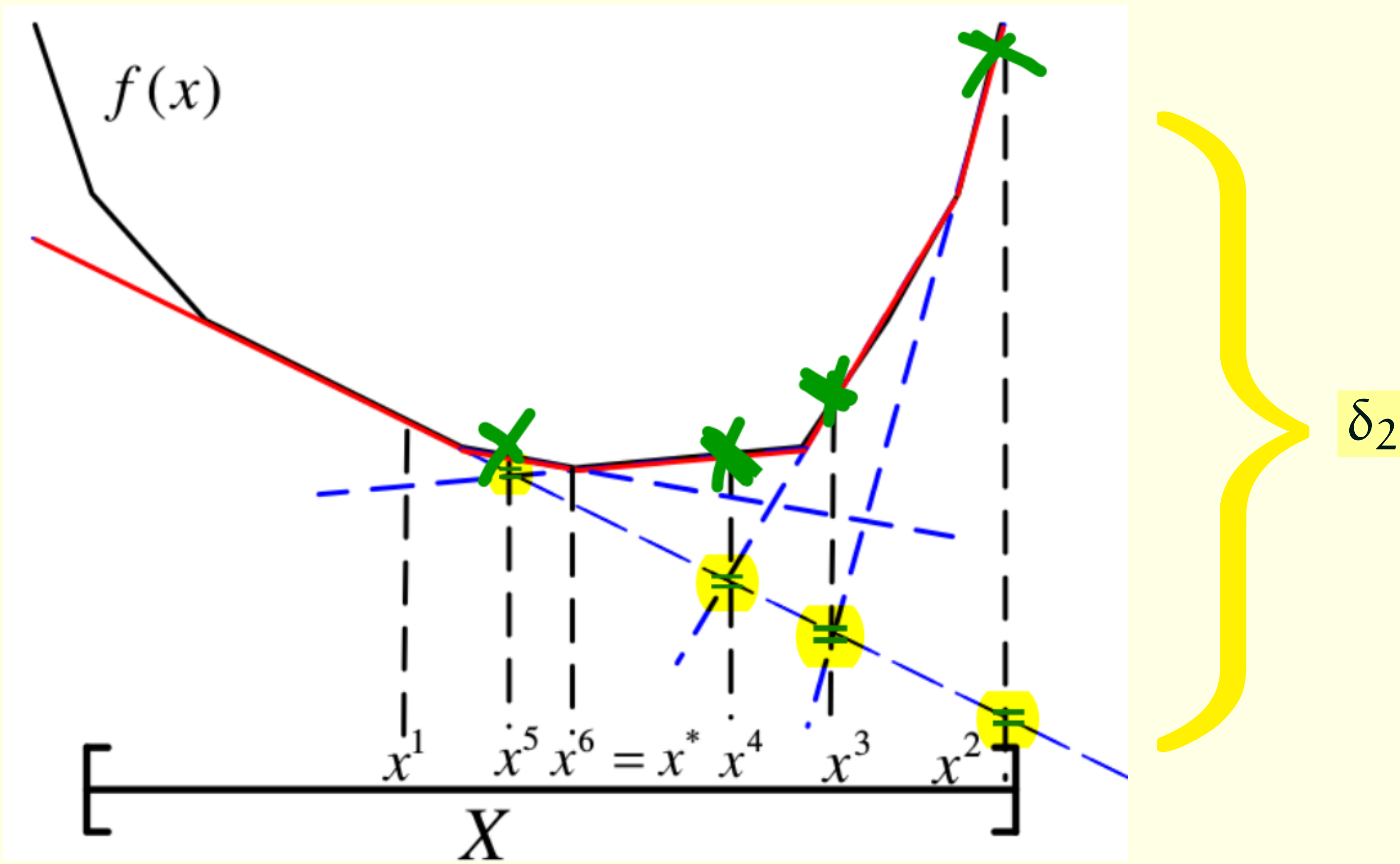

$\{\mathbf{M}_k(x^{k+1})\}$ increases but not necessarily the functional values: $f(x^5) > f(x^4)$. Stopping test measures $\delta_k := f(x^k) - \mathbf{M}_{k-1}(x^k)$

Cutting-plane methods for $\min_{x \in X} f(x)$, X compact polyhedron

0 Choose x^1 and set $k = 1$ and $M_0 \equiv -\infty$.

1 Call the oracle at x^k.

2 Build $\mathbf{M}_k(\cdot) = \max\left(\mathbf{M}_{k-1}(\cdot), f^k + g^{k\top}(\cdot - x^k)\right)$ and compute $x^{k+1} \in \arg\min_X \mathbf{M}_k(x)$

3 Set $k = k+1$ and loop to 1.

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Properties:

1. $\forall k \qquad f^k + g^{k\top}(\cdot - x^k) \leq M_k(\cdot) \leq M_{k+1}(\cdot) \leq f(\cdot)$

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   $\Longrightarrow \{\mathrm{diam}(\partial f(x^k))\} \le M$ Teorema 1.39 Otimização II, Izmailov&Solodov

Cutting-plane methods for $\min_{x \in X} f(x)$, X compact polyhedron

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex and X is convex and compact, and take $\mathrm{tol} = 0$. Then

$$\lim M_k(x^{k+1}) = \liminf f(x^k) = \bar{f}_X = \min_X f(x)$$

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a better recipe

CP methods are

an improvement over subgradient methods

CP methods are like caipirinha with a few drops of cachaça

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can be improved!

Cutting-plane methods: why not the best recipe

$$\begin{cases} \text{Non-monotone functional values, but converges} \\ \text{because } \liminf\left(f^k - \mathbf{M}_{k-1}(x^k)\right) \to 0 \\ \text{Has a stopping test, but LP size grows indefinitely} \\ \text{eventually numerical errors prevail.} \end{cases}$$

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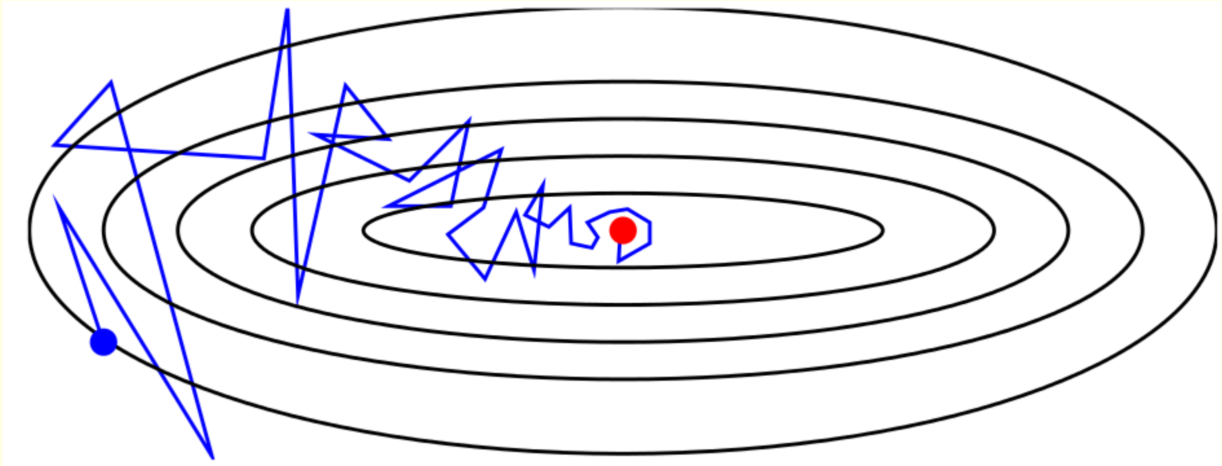
- CP brings in the concept of a model, which gives a stopping test (δ_k)
- CP still non-monotone on f



Monotonicity defeats instability and oscillations

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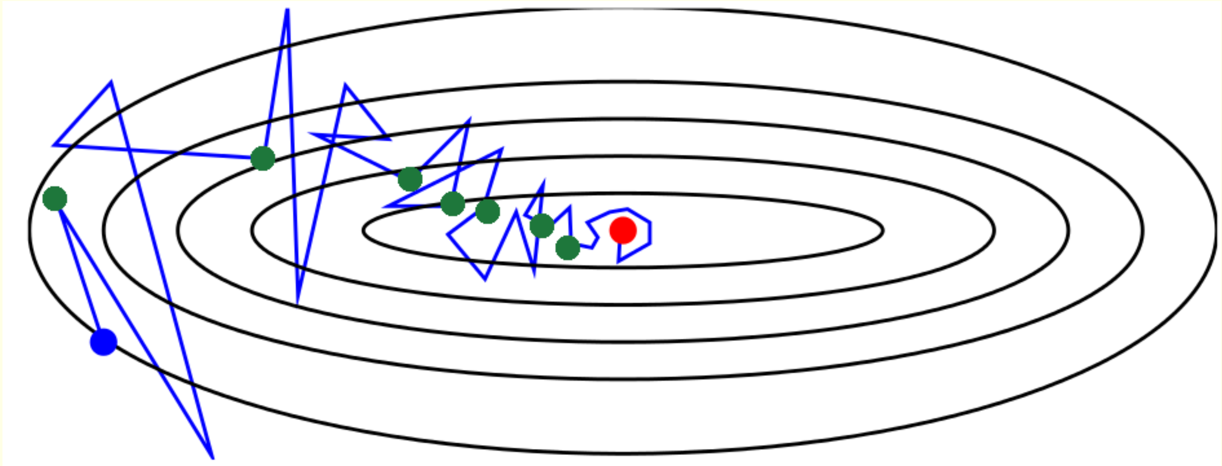

Monotonicity defeats instability and oscillations: the sequence of function values at green-spot iterates converges

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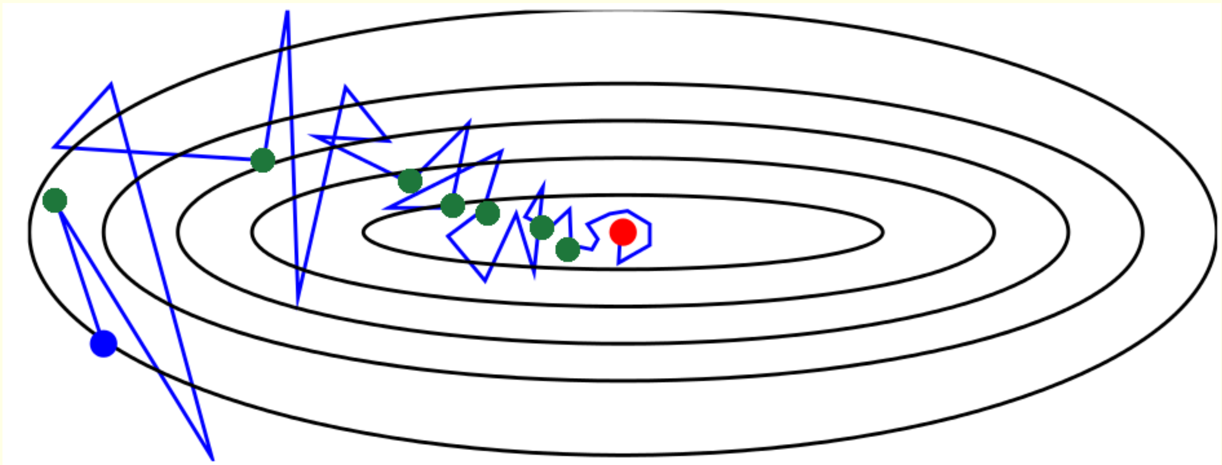

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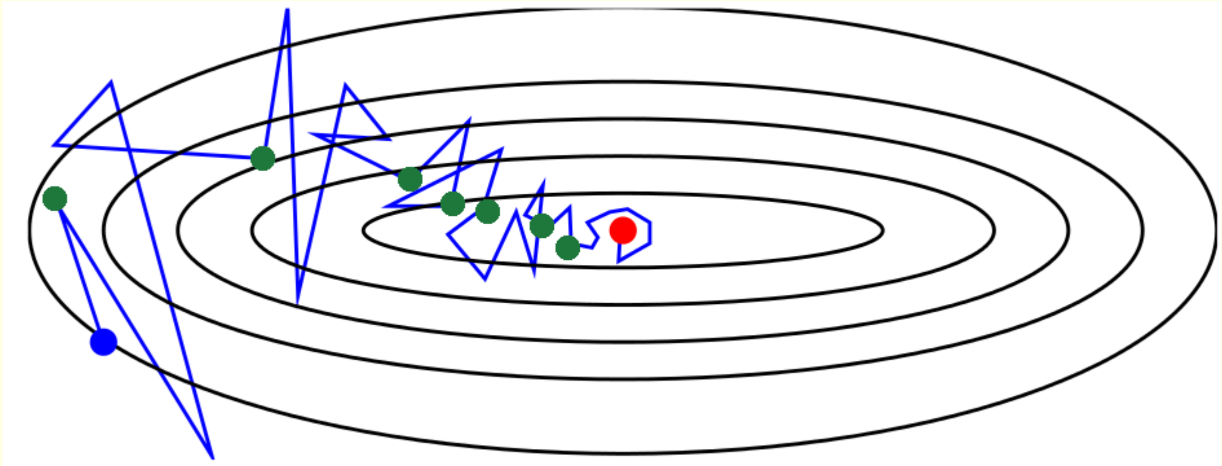

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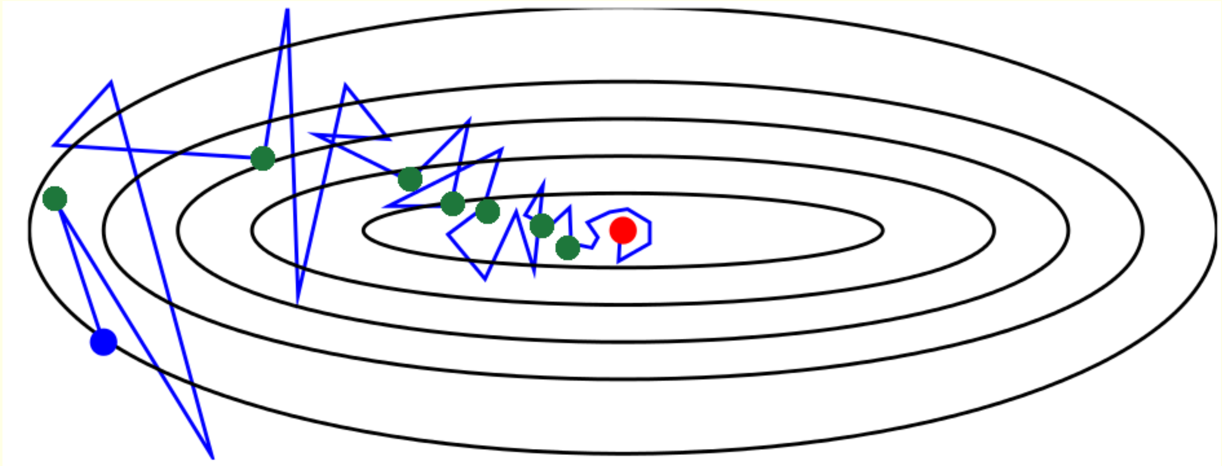

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a good recipe!

Welington de Oliveira, later

Cutting-plane method for 2SLP with fixed recourse (W not random)

is called the

L-shaped method,

by Van Slyke and Wets.

Note: in Integer Programming, same ideas are used for the Benders Decomposition

Two-Stage LP with fixed RCR[a], $\Omega = \{\omega^1, \dots, \omega^S\}$

$$\min_{\mathbf{x}\in\mathbf{X}} \mathbf{c}^\top \mathbf{x} + \phi(\mathbf{x}) \qquad \text{for} \qquad \mathbf{X} := \{\mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} = \mathbf{b}\},$$

where $\phi(x^{\mathbf{k}}) = \mathbb{E}\left[Q(x^{\mathbf{k}}, \xi)\right] = \sum_{s=1}^{S} p_s Q(x^{\mathbf{k}}, \xi^s)$ and

$$Q(x^{\mathbf{k}}, \xi^s) = \left\{ \begin{array}{ll} \min & q^{s\top} y \\ \text{s.t.} & Wy = h^s - T^s x^{\mathbf{k}} \\ & y \geq 0 \end{array} \right. = \left\{ \begin{array}{ll} \max & \pi^\top (h^s - T^s x^{\mathbf{k}}) \\ \text{s.t.} & W^\top \pi \leq q^s \end{array} \right.$$

$$\partial\phi(x^{\mathbf{k}}) = -\sum_{s=1}^{S} p_s T^{s\top} \arg\max \left\{ \pi^\top (h^s - T^s x^{\mathbf{k}}) : \pi \in \Pi(q^s) \right\}$$

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[a]today: without Relative Complete Recourse (infeasibility yields $\phi(x^k) = +\infty$)

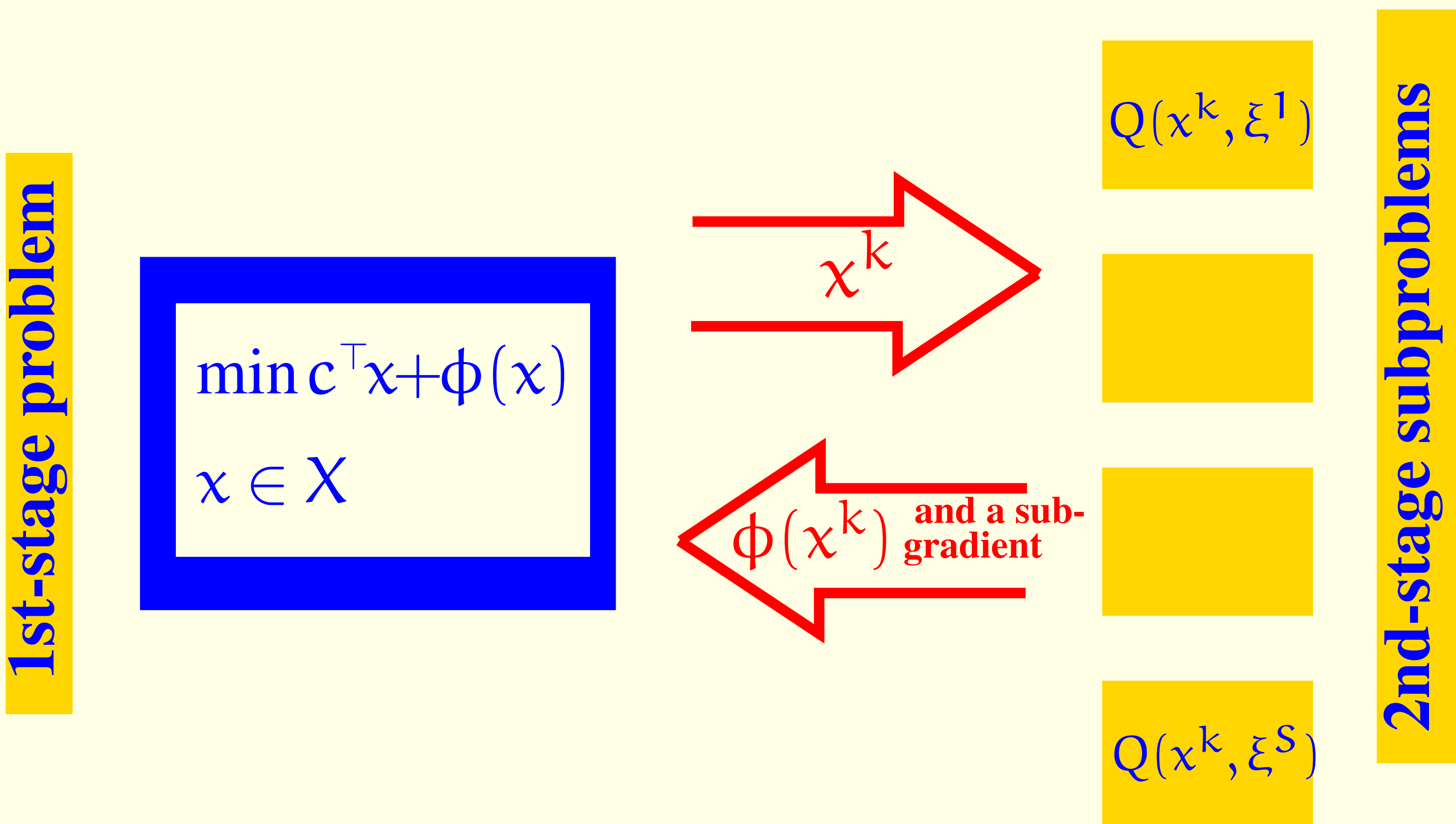
Evaluating $\phi(x^k) = \sum_{s=1}^{S} p_s \pi^{s,k\top}(h^s - T^s x^k)$

gives for free a subgradient $\gamma^k = -\sum_{s=1}^{S} p_s T^{s\top} \pi^{s,k} \in \partial\phi(x^k)$ and

the linearization

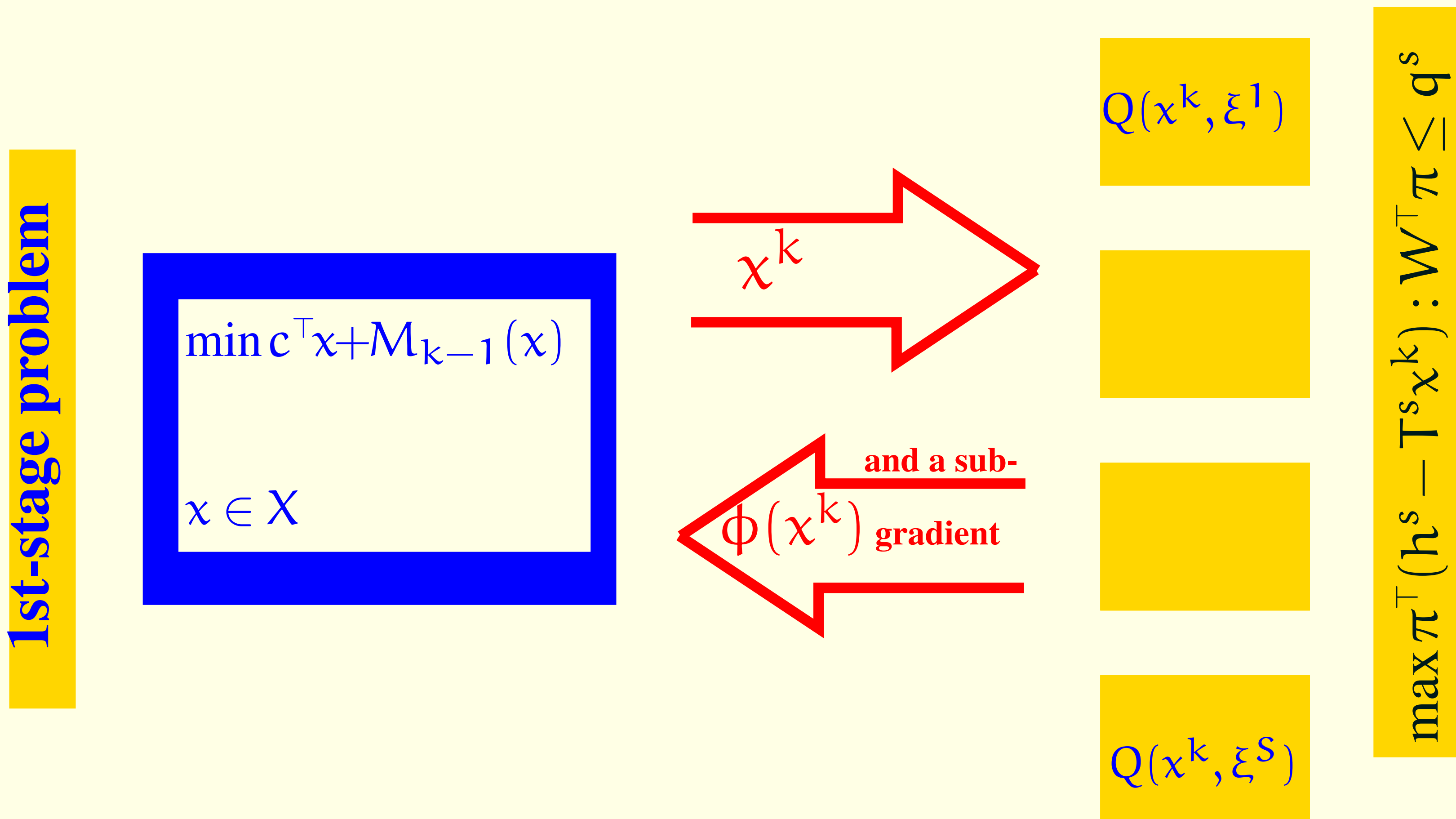
$$\begin{aligned}
\phi(x) &\geq \phi(x^k) + \gamma^{k\top}(x - x^k) \\
&= \sum_{s=1}^{S} p_s \pi^{s,k\top}(h^s - T^s x^k) - \sum_{s=1}^{S} p_s \pi^{s,k\top} T^s (x - x^k) \\
&= \sum_{s=1}^{S} p_s \pi^{s,k\top}(h^s - T^s x)
\end{aligned}$$

Graphically



L-shaped method kth iteration

replaces ϕ by M_{k-1}



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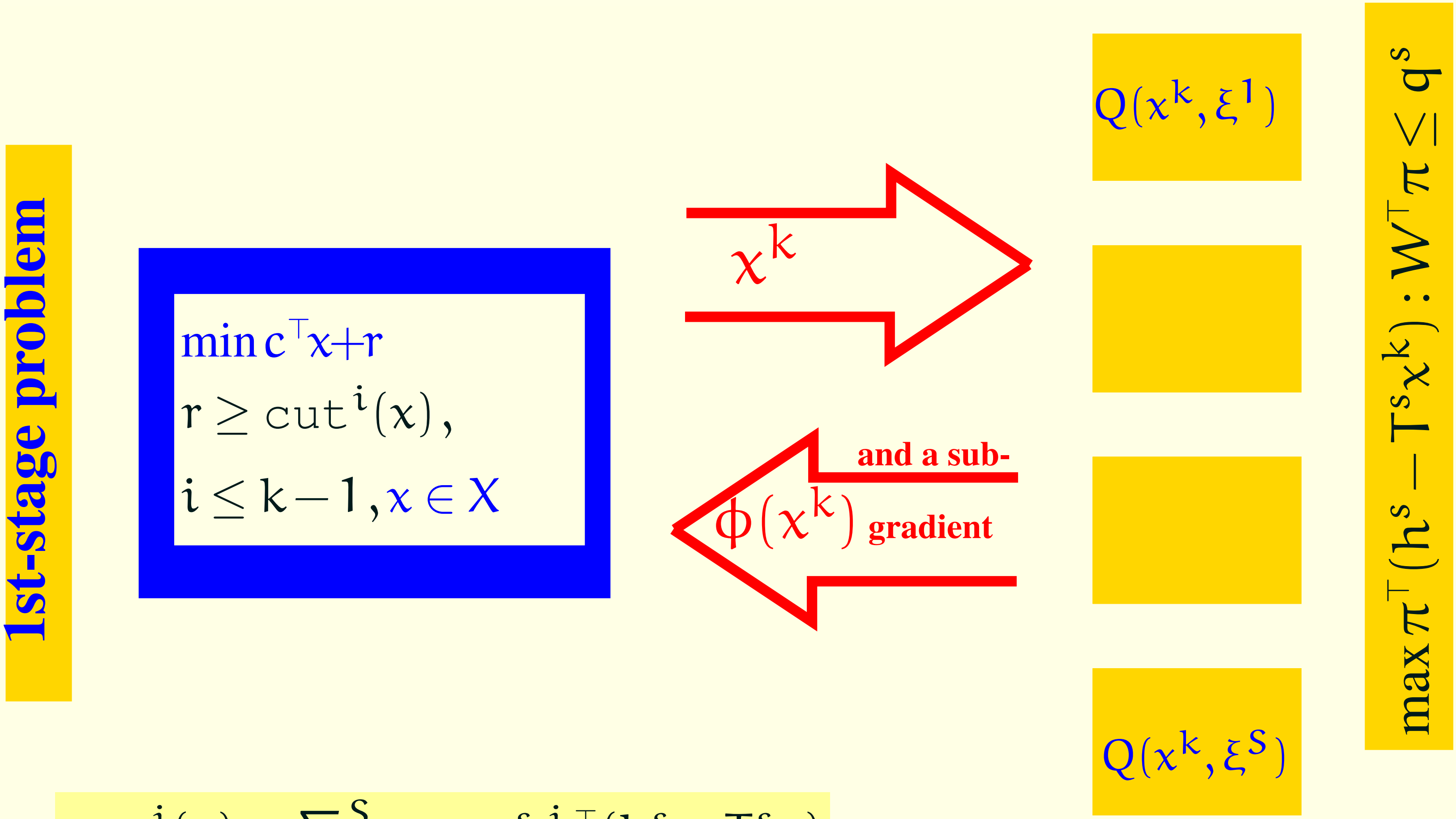
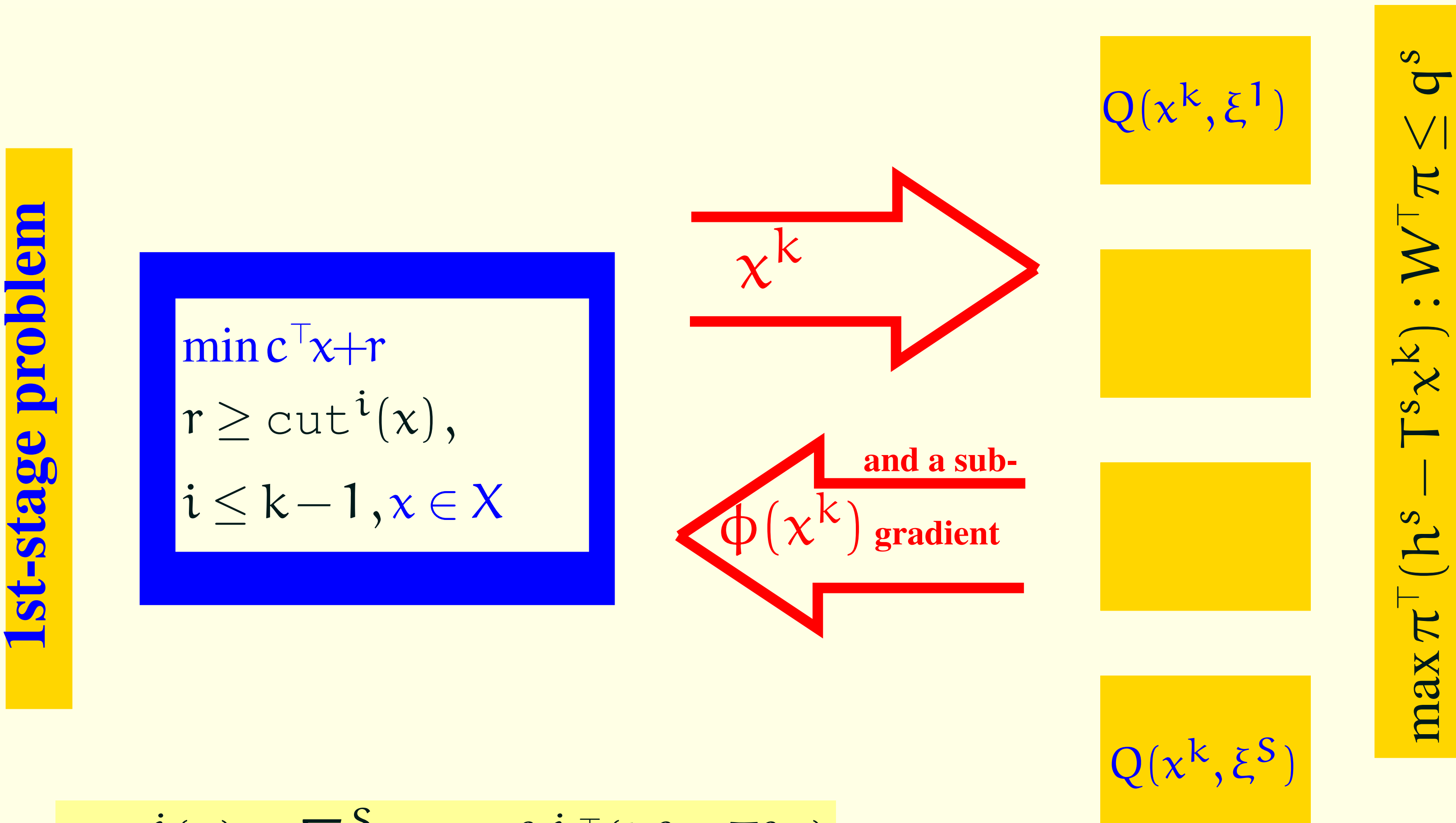


$\mathrm{cut}^i(x) = \sum_{s=1}^{S} p_s \pi^{s,i\,\top} (h^s - T^s x)$

L-shaped method kth iteration

Finite termination if LP solver uses a simplex method (only vertices)



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Finite termination if LP solver uses a simplex method (only vertices)

as long as $\phi : \mathbb{R}^n \to \mathbb{R}$

1st-stage problem

$$\min c^\top x + r$$
$$r \ge \mathrm{cut}^i(x),$$
$$i \le k-1, x \in X$$

x^k

$\phi(x^k)$ **and a sub-gradient**

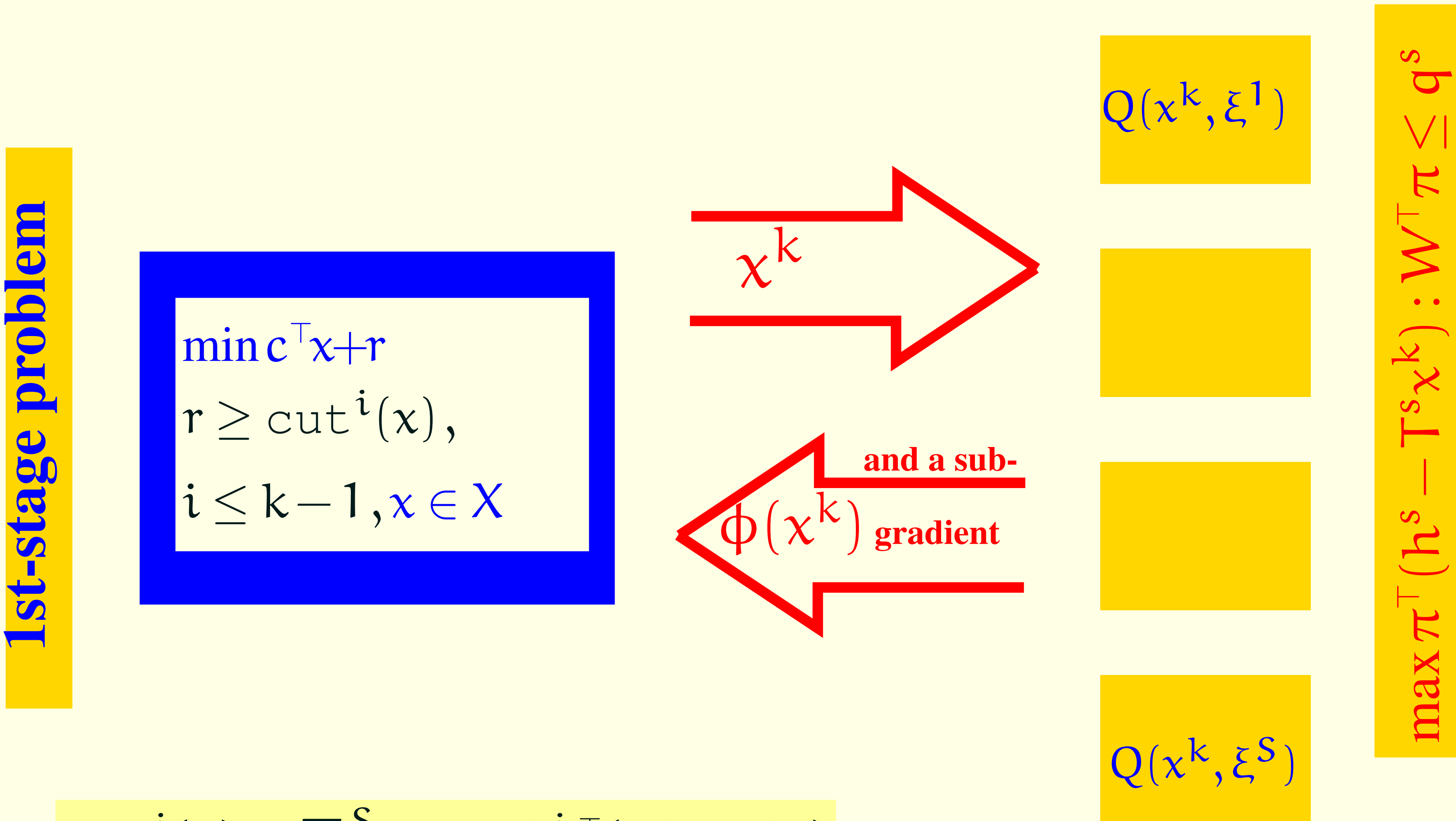
$Q(x^k, \xi^1)$

$Q(x^k, \xi^S)$

$\max \pi^\top (h^s - T^s x^k) : W^\top \pi \le q^s$

$$\mathrm{cut}^i(x) = \sum_{s=1}^{S} p_s \pi^{s,i\,\top}(h^s - T^s x)$$

L-shaped method: what about infeasibility?



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Infeasibility in the primal formulation amounts to dual unboundedness

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$x^k \notin \operatorname{dom} \phi$ (just one $Q(\cdot, \xi^s)$ suffices) makes ϕ extended-valued

L-shaped method: what about infeasibility?

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We'll build **feasibility cuts for each scenario s**

$\mathtt{Feas-cut}^{s,k}(x) = \eta^{s,k\,\top}(h^s - T^s x)$ **for** s **such that** $x^k \notin \operatorname{dom} Q(\cdot, \xi^s)$

L-shaped method: feasibility cuts

– $\operatorname{dom} U(\cdot, \xi^s) = \mathbb{R}^n$ and $U(\cdot, \xi^s)$ is polyhedral.

– $Q(x, \xi^s) < +\infty \iff U(x, \xi^s) = 0$

$$\forall x \in \operatorname{dom} Q(\cdot, \xi^s) \; \mathtt{Feas-Cut}^{s,k}(x) \leq 0$$

– Polyhedral norm, ℓ_1 or ℓ_∞, gives LP (respective dual norms are ℓ_∞ or ℓ_1)

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L-shaped method: convergence

Like we saw for the cutting-plane method for $f : \mathbb{R}^n \to \mathbb{R}$, the method has finite termination:

Now $f(x) = c^\top x + \phi(x)$ is defined on the extended reals, but $\operatorname{epi} Q(\cdot, \xi^s)$ is a closed convex polyhedron and its intersection with the compact X can be characterized by a **finite** number of **basic** objective and feasibility cuts. of the form

$$\begin{aligned} \texttt{Obj}-\texttt{cut}^i(x) &= \textstyle\sum_{s=1}^S p_s \pi^{s,i\,\top}(h^s - T^s x) & \text{if } x^i \in \operatorname{dom} \phi \\ \texttt{Feas}-\texttt{cut}^{s,i}(x) &= \eta^{s,i\,\top}(h^s - T^s x) & \text{if } x^i \notin \operatorname{dom} Q(\cdot, \xi^s) \end{aligned}$$

L-shaped method kth iteration

$$\begin{aligned} \texttt{Obj-cut}^i(x) &= \textstyle\sum_{s=1}^S p_s \pi^{s,i\top}(h^s - T^s x) && \text{if } x^i \in \operatorname{dom}\phi \\ \texttt{Feas-cut}^{s,i}(x) &= \eta^{s,i\top}(h^s - T^s x) && \text{if } x^i \notin \operatorname{dom} Q(\cdot,\xi^s) \end{aligned}$$

Let

$$\begin{aligned} J^k_{\texttt{Obj}} &= \{i < k : x^i \in \operatorname{dom}\phi\} \\ \text{and, for } s = 1,\ldots,S \quad J^{s,k}_{\texttt{Feas}} &= \{i < k : x^i \notin \operatorname{dom} Q(\cdot,\xi^s)\} \end{aligned}$$

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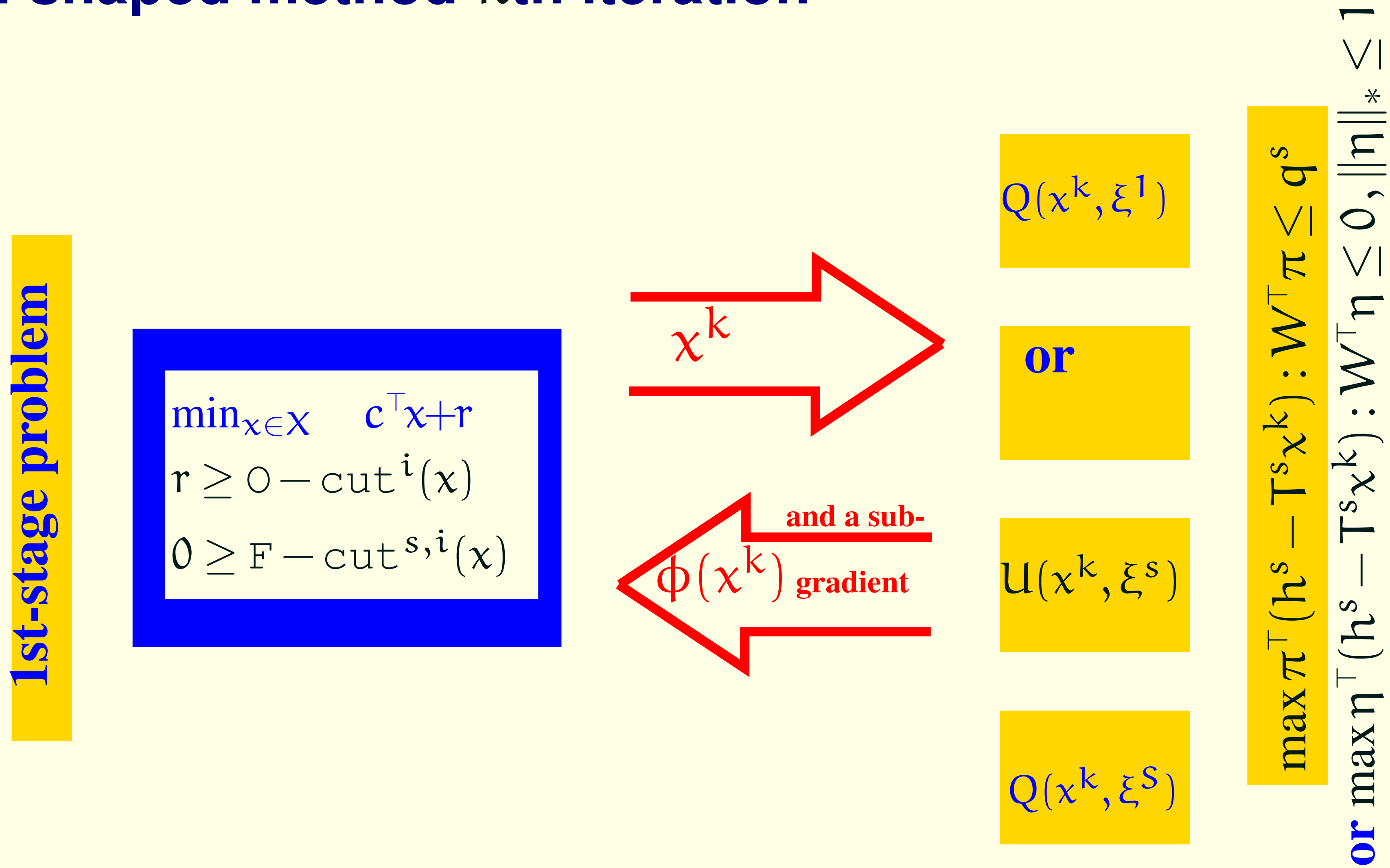
Then the 1st-stage problem has the form

$$\left\{\begin{array}{lll} \min & c^\top x + r & \\ \text{s.t.} & r \geq \texttt{O-cut}^i(x) & \text{for } i \in J^{k-1}_{\mathrm{Obj}} \\ & 0 \geq \texttt{F-cut}^{s,i}(x) & \text{for } i \in J^{s,k-1}_{\mathrm{Feas}} \text{ and } s = 1, \ldots, S \\ & x \in X & \end{array}\right.$$

L-shaped method kth iteration



L-shaped method kth iteration

1st-stage problem

$$\min_{x \in X} \quad c^\top x + r$$

$$r \geq 0 - \mathrm{cut}^i(x)$$

$$0 \geq F - \mathrm{cut}^{s,i}(x)$$

x^k

$\phi(x^k)$ and a sub-gradient

$Q(x^k, \xi^1)$

or

$U(x^k, \xi^s)$

$Q(x^k, \xi^S)$

$\max \pi^\top (h^s - T^s x^k) : W^\top \pi \leq q^s$

or $\max \eta^\top (h^s - T^s x^k) : W^\top \eta \leq 0, \|\eta\|_* \leq 1$

there is also a multi-cut variant