Risk-constrained dynamic asset allocation
via
stochastic dual dynamic programming

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Objective and contributions

Develop a realistic and computationally tractable stochastic dynamic asset allocation model considering time consistent and intuitive risk constraints, time dependence and transaction cost.

Main Contributions:

• Time consistent risk-constrained stochastic dynamic asset allocation model
  – Realistic: transaction costs and time dependence
  – Risk aversion: intuitive user defined loss limit
  – Computationally tractable: SDDP with Markovian policy
No transactional cost and time independence

\[ Q_t(W_t) = \max_{a_t, c_t, d_t \geq 0} \phi \left[ Q_{t+1} \left( c_t + \sum_{i \in \mathcal{A}} (1 + r_{i,t+1}) a_{i,t} \right) \right] \]

s. t. \[ c_t + \sum_{i \in \mathcal{A}} a_{i,t} = W_t \]

Where, \( Q_T(W_T) = W_T \)
Time consistent (recursive) model

- Usually the risk measure is the convex combination of the expect return and the CVaR

\[ \phi(W_t) = (1 - \lambda)E[W_t] + \lambda CVaR_\alpha(W_t) \]

- Economic interpretation: certain equivalent
  - Rudloff, Street, Valladão (2014)
- Problem: How should we define \( \lambda \)?
Risk constrained model

\[ V_t(W_t) = \max_{a_t, c_t \geq 0} \mathbb{E} \left[ V_{t+1} \left( c_t + \sum_{i \in \mathcal{A}} (1 + r_{i,t+1}) a_{i,t} \right) \right] \]
\[ s.t. \quad \rho_t \left[ \sum_{i \in \mathcal{A}} r_{i,t+1} a_{i,t} \right] \leq \gamma W_t \]
\[ c_t + \sum_{i \in \mathcal{A}} a_{i,t} = W_t \]

• Intuitive risk averse parameter \( \gamma \)
• Relative complete recourse for \( \gamma \geq 0 \).
  – One can always allocate in cash
• Positively homogeneous

If \( W_t \geq 0 \), \( V_t(W_t) = W_t \cdot V_t(1) \)
Myopic Solution

\[ V_t(W_t) = \max_{a_t, c_t \geq 0} \mathbb{E} \left[ V_{t+1} \left( c_t + \sum_{i \in \mathcal{A}} (1 + r_{i,t+1})a_{i,t} \right) \right] \]

s.t.

\[ \rho_t \left[ \sum_{i \in \mathcal{A}} r_{i,t+1}a_{i,t} \right] \leq \gamma W_t \]

\[ c_t + \sum_{i \in \mathcal{A}} a_{i,t} = W_t \]
Myopic Solution

\[ V_t(W_t) = \max_{a_t, c_t \geq 0} \left[ c_t + \sum_{i \in \mathcal{A}} (1 + r_{i,t+1}) a_{i,t} \right] \]

s.t.

\[ \rho_t \left[ \sum_{i \in \mathcal{A}} r_{i,t+1} a_{i,t} \right] \leq \gamma W_t \]

\[ c_t + \sum_{i \in \mathcal{A}} a_{i,t} = W_t \]
Transactional cost and time independence

\[ V_t(x_t) = \max_{a_t, c_t, d_t \geq 0} \mathbb{E}[V_{t+1}(x_{t+1})] \]

s.t.

\[ \rho_t \left[ \sum_i \left( r_{i,t+1} a_{i,t} - \tau (d_{i,t}^+ + d_{i,t}^-) \right) \right] \leq \gamma \left( x_t^c + \sum_i x_{i,t}^a \right) \]

\[ a_{i,t} = x_{i,t}^a + d_{i,t}^+ - d_{i,t}^-, \forall i \in \mathcal{A} \]

\[ c_t = x_t^c - (1 + \tau) \sum_i d_{i,t}^+ + (1 - \tau) \sum_i d_{i,t}^- \]

\[
\begin{bmatrix}
  a_{t-1} \\
  c_{t-1}
\end{bmatrix}
\begin{bmatrix}
  r_t \\
  c_t
\end{bmatrix}
\begin{bmatrix}
  a_t \\
  c_t
\end{bmatrix}
\]

\[ x_{t+1} = \begin{bmatrix}
  x_{t+1}^a \\
  x_{t+1}^c \\
  c_{t+1}
\end{bmatrix} = \begin{bmatrix}
  a_t (1 + r) \\
  x_t^c (1 + r_f) \\
  c_t (1 + r_f)
\end{bmatrix} \]
Simplifying notation

\[ V_T(x_T) = x_T^\xi + \sum_i x_{i,t}^a \]

\[ V_t(x_t) = \max_{u \in U(x_t)} \mathbb{E}[V_{t+1}(x_{t+1}(u))], \quad \forall t \in \{0, \ldots, T - 1\} \]

\[ U(x) = U((x^c, x^a)) \]

\[
\begin{aligned}
&= \left\{ (c, a, d^+, d^-) \in \mathbb{R}_{+}^{3N+1} \mid \\
&\quad \rho_t \left[ \sum_i (r_{i,t+1} a_{i,t} - \tau (d_{i,t}^+ + d_{i,t}^-)) \right] \leq \gamma (x_T^\xi + \sum_i x_{i,t}^a) \right. \\
&\quad c = x^c - (1 + \tau) \sum_{i \in A} d_i^+ + (1 - \tau) \sum_{i \in A} d_i^- \\\n&\quad a_i = x_{i,t}^a + d_i^+ - d_i^-, \quad \forall i \in A
\end{aligned}
\]

\[ u = (c, a_1, \ldots a_N, d_1^+, \ldots, d_N^+, d_1^-, \ldots, d_N^-)' \]

\[ W_T(u) = W_T(c, a) = c + \sum_{i \in A} (1 + r_{i,T})a_i \]

\[ x_{t+1}(u) = x_{t+1}(c, a) = (c_t, (1 + r_{1,t+1})a_{1,t}, \ldots, (1 + r_{N,t+1}) a_{N,t})' \]
Transaction cost and time independence

- Solution algorithm: SDDP

\[
V_t(x_t) = \max_{u \in U(x_t)} \mathbb{E}[V_{t+1}(x_{t+1}(u))], \quad \forall t \in \{0, ..., T - 1\}
\]
Transaction cost and time dependence

\[ V_t(x_t, r_{[t]}) = \max_{u \in U(x_t)} \mathbb{E}[V_{t+1}(x_{t+1}(u)|r_{[t]}], \quad \forall t \in \{0, \ldots, T-1\} \]
Transactional cost and time dependence

\[ V_t(x_t, r_{[t]}) = \max_{u \in U(x_t)} \mathbb{E}[V_{t+1}(x_{t+1}(u)|r_{[t]}], \quad \forall t \in \{0, ..., T - 1\} \]
Transactional cost and Markov dependence

\[ V_t^k(x_t) = \max_{u \in U(x_t)} \sum_{j=1}^{K} \mathbb{E}[V_{t+1}^k(x_{t+1}(u))|S_{t+1} = j] \mathbb{P}(S_{t+1} = j|S_t = k) \]

![Diagram showing the states and transitions between Bull, Bear, and Neutral, with probabilities for transitions labeled.]
Transactional cost and Markov **dependence**

\[ \mathbb{P}(S_t = 1 | S_{t-1} = k) + \mathbb{P}(S_t = 2 | S_{t-1} = k) \]

\[ \mathbb{P}(S_t = 2 | S_{t-1} = k) \]
Transactional cost and Markov **dependence**

\[
P(S_t = 1 | S_{t-1} = k) + P(S_t = 2 | S_{t-1} = k)
\]
Transactional cost and Markov dependence

\[
\mathbb{P}(S_t = 1 | S_{t-1} = k) + \mathbb{P}(S_t = 2 | S_{t-1} = k)
\]
Case of study

• Five types of investments: gold, IBOV, dollar and euro
• 4 states for HMM, 0.05% transactional costs, $\alpha = 0.9$ and $\gamma = 0.1$
• Uses 2 years with sliding window to evaluate the model, starting in 2008
• Simulate on out of sample data with 10 stages (days)
• Besides the cumulative performance we also evaluate the performance for 6 consecutive months
Cumulative portfolio return
Portfolio trailing return

Equal
Myopic
SDDP
Portfolio allocation
Conclusions

We presented a risk-constrained dynamic asset allocation model with transaction cost and time dependence assuming a Markov Model for asset returns.

Main Contributions:
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  – Risk aversion: intuitive user defined loss limit
  – Computationally tractable: SDDP with Markovian policy
Thank you!!!

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Stochastic Dynamic Dual Programming
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RHS dependence

\[
Q_t(x_{t-1}, \xi_t) = \min_{x_t} c_t^\top \cdot x_t + Q_{t+1}(x_t) \\
\text{s.t. } A_t \cdot x_t = -B_t x_{t-1} + b_t \\
x_t \geq 0
\]

\[
b_t = \phi b_{t-1} + \varepsilon_t
\]

\[
Q(x_{t-1}, b_{t-1}, \xi_t) = \min_{x_t} c_t^\top \cdot x_t + Q_{t+1}(x_t) \\
\text{s.t. } A_t x_t = -B_t x_{t-1} + \phi_t b_{t-1} + \varepsilon_t \\
x_t \geq 0
\]