On de Finetti's problem under a time of ruin constraint

Spectrally negative reserves with completely monotone Lévy measure

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Outline

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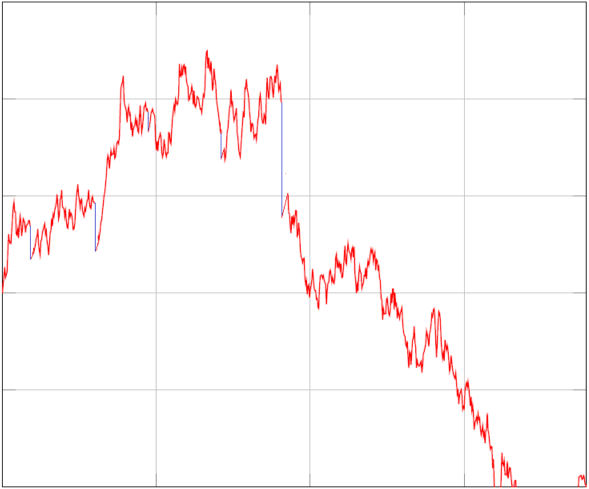
  Available resources by the insurance company. Claims are modeled with negative jumps.



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  Adapted, non-negative, non-decreasing, *càglàd* processes with $D_0 = 0$.
- Surplus Process, $X^D := X - D$:
  Process D cannot lead to ruin, so $D_{t+} - D_t \leq X_t^D \vee 0$.
  Let Θ be the set of such strategies.

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De Finetti's problem [De Finetti, 1957] consists in finding

$$\sup_{D \in \Theta} \mathcal{V}^D(x_0).$$

The solution to this problem has been extensively studied for different classes of reserve processes ([Schmidli, 2008], [Loeffen, 2008].)

Motivation

There exists a trade-off between stability and profitability. The alternative problem of minimizing ruin probability means no dividend payment and profits are 0. On the contrary, maximizing the dividends leads to a dividend payment trend for which ruin is certain regardless of the initial amount x_0.

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Some problems have been proposed to deal with this trade-off: [Paulsen, 2003],[Thonhauser and Albrecher, 2007], [Loeffen and Renaud, 2010].

The Problem

Inspired by previous work of [Thonhauser and Albrecher, 2007] we introduce a constraint on the time of ruin and state the following problem:

$$V(x) := \sup_{D\in\Theta} \mathcal{V}^D(x) \tag{P}$$
$$\text{s.t.} \quad \mathbb{E}_x\left[\int_0^{\tau^D} e^{-qs}ds\right] \geq \int_0^{T} e^{-qs}ds \quad T \text{ fixed},$$

Define $K_T := \int_0^T e^{-qs}ds$.

In order to solve this problem we use Lagrangian relaxation to reformulate our problem. We first define the following function

$$\mathcal{V}_\Lambda^D(x) := \mathbb{E}_x\left[\int_0^{\tau^D} e^{-qt} dD_t + \Lambda \int_0^{\tau^D} e^{-qs} ds\right] - \Lambda K_T \qquad \text{(L)}$$

$$\text{for} \qquad \Lambda \geq 0.$$

Then, (P) is equivalent to $\sup_{D \in \Theta} \inf_{\Lambda \geq 0} \mathcal{V}_\Lambda^D(x)$ since

$$\inf_{\Lambda \geq 0} \mathcal{V}_\Lambda^D(x) = \begin{cases} \mathcal{V}^D(x) & \text{if } \mathbb{E}_x\left[\int_0^{\tau^D} e^{-\delta s} ds\right] \geq K_T \\ -\infty & \text{otherwise .} \end{cases}$$

Dual problem

The dual problem of (P) is defined as

$$\inf_{\Lambda \geq 0} \sup_{D \in \Theta} \mathcal{V}_{\Lambda}^{D}(x). \tag{D}$$

Remember that (P) $\leq$ (D) always holds.
Therefore, once (D) is solved, the main goal is to prove that there is no duality gap, that is

$$\sup_{D \in \Theta} \inf_{\Lambda \geq 0} \mathcal{V}_{\Lambda}^{D}(x) = \inf_{\Lambda \geq 0} \sup_{D \in \Theta} \mathcal{V}_{\Lambda}^{D}(x).$$

Recall

$$\mathcal{V}_{\Lambda}^{D}(x) = \mathbb{E}_x\left[\int_0^{\tau^D} e^{-qt}dD_t + \Lambda\int_0^{\tau^D} e^{-qt}dt\right] - \Lambda K_T.$$

To solve (D), we can use previous results on de Finetti's problem and compute

$$V_{\Lambda}(x) := \sup_{D\in\Theta} \mathcal{V}_{\Lambda}^{D}(x) \tag{Pa}$$

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Where to start? A simple scenario.

Assume the reserves process follow the Cramér-Lundberg model with exponentially distributed claim sizes.

Theorem

Let $x \geq 0$ and $V(x)$, $V_\Lambda(x)$ be the optimal solutions to (P), (Pa), *respectively. Then, strong duality holds, i.e.,*

$$\inf_{\Lambda \geq 0} V_\Lambda(x) = V(x).$$

- Key point: **Barrier strategies are optimal.**

[Hernández and Junca, 2015]

Can we do more?

So far, we succeeded to solve (P) under the particular case of reserves that follow the Cramér-Lundberg model with claims that are exponentially distributed.

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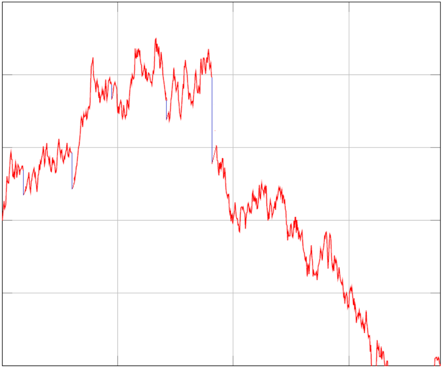
But how about more general reserves processes?



Some definitions

- X is a *spectrally negative Lévy process*, SNLP, if its Lévy triple (a, σ, Π) is such that Π is supported on $(-\infty, 0)$, i.e., they have no positive jumps, and do not have monotone paths a.s.

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- For SNLP we define the *Laplace exponent* and its inverse by

$$\psi(\theta) := \log(\mathbb{E}[e^{\theta X_1}]) \quad \theta \geq 0,$$
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- There exist a family of functions called q-scale $W^{(q)} : \mathbb{R} \to [0, \infty)$ defined for each $q \geq 0$ s.t. $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is a strictly increasing and continuous function on $[0, \infty)$ whose Laplace transform satisfies,

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q} \qquad \text{for } \beta > \Phi(q). \qquad (1)$$

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- For each $q \geq 0$ define $Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(z) dz$, for $x \in \mathbb{R}$.

Consider a barrier strategy at level b, i.e., $D_t^b := (b \vee \overline{X}_t) - b$, $t \geq 0$, so $X_t^{D^b} = b - [(b \vee \overline{X}_t) - X_t]$. This motivates the following: For a Lévy process Y the *reflected process at its supremum with initial value s* is defined as

$$\hat{Y}_t^s := (s \vee \overline{Y}_t) - Y_t, \quad t \geq 0. \tag{2}$$

For such processes define also the exit time

$$\hat{\sigma}_k^s := \inf\{t > 0 : \hat{Y}_t^s > k\}. \tag{3}$$

Therefore,

$$X_t^{D^b} = b - \hat{X}_t^b$$

and

$$\tau^b := \tau^{D^b} = \hat{\sigma}_b^b.$$

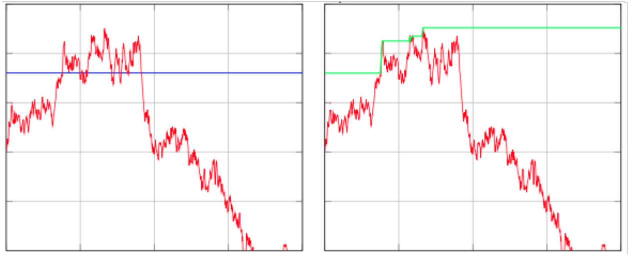

Figure : Process X_t and process $(b \vee \overline{X}_t)$ (green line)

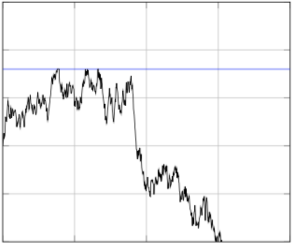

Figure : $X_t^{D^b} = b - \hat{X}_t^b$

The following results are the link between (D) and the q-scale functions:

Theorem ([Gerber, 1972])

Let $b > 0$ and consider the process D^b as before. For $x \in [0, b]$,

$$\mathbb{E}_x \left[\int_0^{\tau^b} e^{-qs} dD^b_s \right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(b+)}, \tag{4}$$

where $W^{(q)\prime}(b+)$ is understood as the right derivative of $W^{(q)}$ at b.

Theorem ([Avram et al., 2004])

Let $b > 0$ and consider the process D^b as before. For $x \in [0, b]$,

$$\mathbb{E}_x\left[e^{-q\tau^b}\right] = Z^{(q)}(x) - q\frac{W^{(q)}(x)}{W^{(q)\prime}(b)}W^{(q)}(b).$$

Therefore,

$$\mathbb{E}_x\left[\int_0^{\tau^b} e^{-qs}ds\right] = \frac{W^{(q)}(x)}{W^{(q)\prime}(b)}W^{(q)}(b) - \int_0^x W^{(q)}(z)dz. \quad (5)$$

(L) for a barrier strategy

Expression (5) and (4) yields the following result.

Proposition ([Loeffen, 2009])

For a sufficiently smooth q-scale function $W^{(q)}$, the value of (L) *for the barrier strategy at level $b \geq 0$ is given by*

$$\mathcal{V}_{\Lambda}^{D^b}(x) = \begin{cases} W^{(q)}(x)\left[\frac{1+\Lambda W^{(q)}(b)}{W^{(q)\prime}(b)}\right] - \Lambda \int_0^x W^{(q)}(z)dz - \Lambda K_T & \text{if} \quad x \leq b \\ x - b + \mathcal{V}_{\Lambda}^{D^b}(b) & \text{if} \quad x > b. \end{cases} \tag{6}$$

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Let $\zeta_\Lambda : [0,\infty) \to \mathbb{R}$ be defined by

$$\zeta_\Lambda(x) := \frac{1 + \Lambda W^{(q)}(x)}{W^{(q)\prime}(x)}, \quad x > 0 \tag{7}$$

and $\zeta_\Lambda(0) := \lim_{x \downarrow 0} \zeta_\Lambda(x)$. In [Loeffen, 2009] it is shown that the barrier strategy at level

$$b^*_\Lambda := \sup\{b : \zeta_\Lambda(b) \geq \zeta_\Lambda(x),\ \text{for all } x \geq 0\} \tag{8}$$

is the optimal strategy for (Pa). Furthermore, when the Lévy measure Π has completely monotone density, the function ζ_Λ has a unique maximum.

Completely monotone functions

Definition

A function f is said to be *completely monotone* if $f \in C[0,\infty)$, $f \in C^{\infty}(0,\infty)$ and satisfies $(-1)^n \frac{d^n}{dx^n} f(x) \geq 0$

Example

$$\frac{1}{(\lambda+\mu x)^{\nu}}, \qquad Ln\Big(b+\frac{c}{x}\Big), \qquad e^{\frac{a}{x}}, \qquad \frac{Ln(1+x)}{x},$$

where $\mu, \lambda, \nu \geq 0$ with no both λ and μ equal to zero and $b \geq 1, a, c > 0$.

Value Function of (Pa)

Theorem ([Loeffen, 2009])

Suppose the Lévy measure of the spectrally negative Lévy process X with Lévy triple (γ, σ, Π), has a completely monotone density. Let $c = \gamma + \int_0^1 x\Pi(dx)$. Then the following holds:

(i) *If $\sigma = 0$ and $\Pi(0,\infty) < \infty$ and $\Lambda < -c/q$ the take-the-money-and-run strategy is an optimal strategy for* (Pa).

(ii) *if $\sigma > 0$, or $\Pi(0,\infty) = \infty$, or $\Pi(0,\infty) < \infty$ and $\Lambda \geq -c/q$, then the optimal strategy consists of a barrier strategy at level b^*_Λ given by equation* (8) *and the corresponding value function is given by equation* (6).

Proposition

For each $b \in (b_0^*, \infty)$ *there exists a* Λ *such that,* D^b*, the barrier strategy at level* b*, is optimal for* (Pa) *with* Λ*.*

Theorem

Let $x \geq 0$*,* $K_T \geq 0$*, and* $V(x)$*,* $V_\Lambda(x)$ *be the optimal solution to* (P), (Pa)*, respectively, with initial value* x*, then, strong duality holds, i.e.,*

$$\inf_{\Lambda \geq 0} V_\Lambda(x) = V(x).$$

Proof. Let τ_b be the time of ruin of the barrier strategy with level b and initial value $x \in [0, b]$. Let

$$\Psi_b(x) = \mathbb{E}_x \left[\int_0^{\tau_b} e^{-qs} ds \right].$$

To obtain the optimal value function of (P) we need to show (b^*, Λ^*) that certify strong duality. To do so we distinguish the following 4 cases depending on the initial value x and the parameter T:

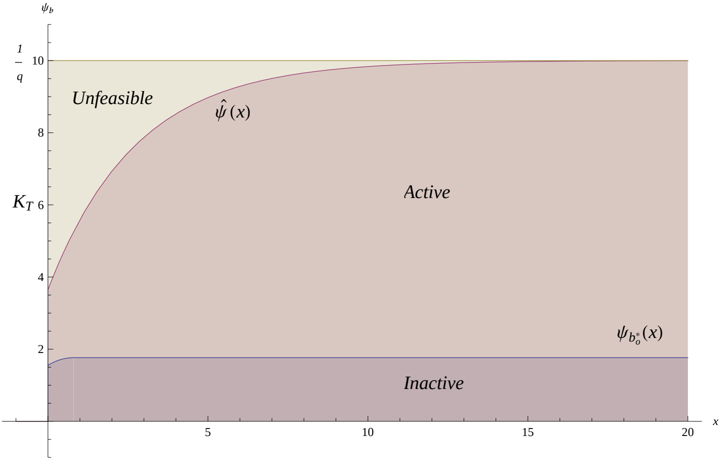

Figure : Characterization of the solution of (P).

Numerical Example

We illustrate the previous result for X with Lévy triple $\left(10, 0.1, \frac{dx}{|x|^{1.8}}1_{(-\infty,0)}\right)$, that is an $0.8-$stable process with diffusion.

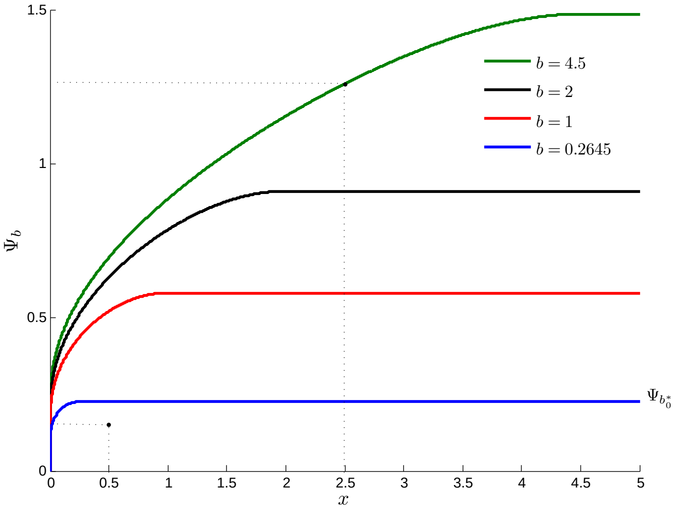

Figure : $\Psi_b(x)$ for different barrier levels. Inactive and biding constraint cases.

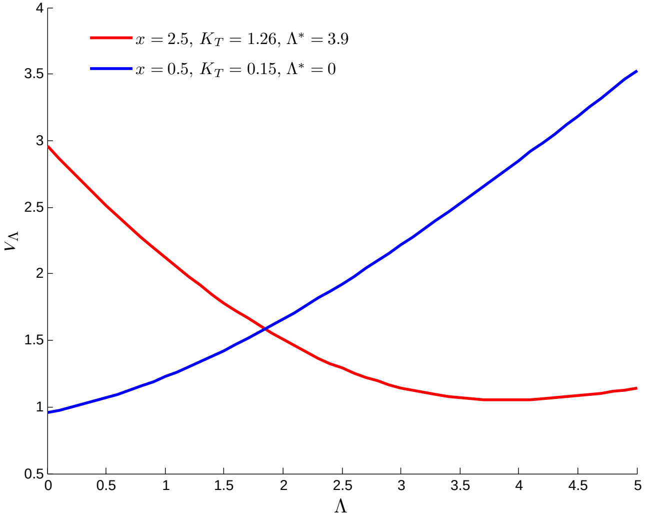

Figure : $V_\Lambda(x)$ for different initial values. Inactive and biding constraint cases.

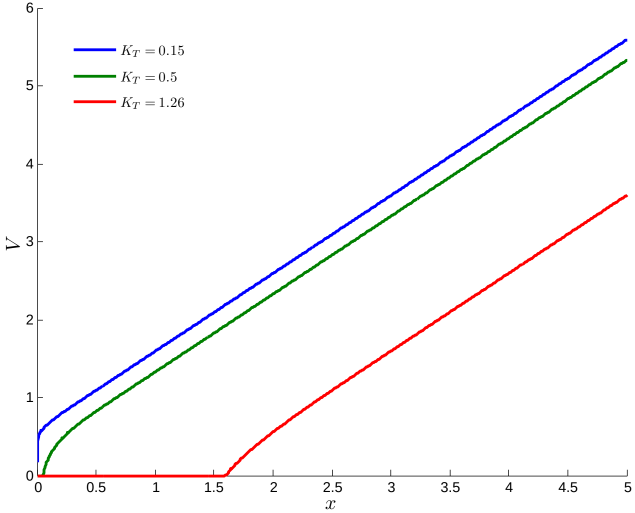

Figure : Value functions for constrained and unconstrained problems for different values of K_T.

Extensions

- Dual model (spectrally positive Levy Processes).
- Optimal dividend with fixed transaction cost. (Impulse control and (S, s) strategy).
- Study other types of restrictions such that:

$$\mathbb{E}_x[\tau^D] \geq T$$

$$\mathbb{P}(\tau^D \leq T | X_0^D = x_0) \leq \epsilon$$

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Thank You