On de Finetti’s problem under a time of ruin constraint
Spectrally negative reserves with completely monotone Lévy measure

Mauricio Junca
Joint with: Camilo Hernández

Universidad de los Andes, Colombia.

AASS Workshop
SVAN 2016
March 29, 2016
Outline

De Finetti’s Problem

De Finetti’s problem under a time of ruin constraint
  Problem formulation
  The Dual Problem

Exponential claims case

Spectrally Negative Lévy reserves
  Solution of dual problem
  Solution of the problem

Numerical example

Extensions
De Finetti’s Problem

De Finetti’s problem under a time of ruin constraint
   Problem formulation
   The Dual Problem

Exponential claims case

Spectrally Negative Lévy reserves
   Solution of dual problem
   Solution of the problem

Numerical example

Extensions
One agent: Insurance company.
- One agent: Insurance company.
- Reserves Process, $X = (X_t)_{t \geq 0}$: Available resources by the insurance company. Claims are modeled with negative jumps.

![Graph](image.png)
One agent: Insurance company.

Reserves Process, $X = (X_t)_{t \geq 0}$: Available resources by the insurance company. Claims are modeled with negative jumps.

Dividend Process, $D = (D_t)_{t \geq 0}$: The company has to decide a dividend payment strategy $D$, which represents the cumulative payments. Adapted, non-negative, non-decreasing, càglàd processes with $D_0 = 0$. 

Surplus Process, $X_D := X - D$: Process $D$ cannot lead to ruin, so $D_t + -D_t \leq X_D_t \vee 0$. Let $\Theta$ be the set of such strategies.
One agent: Insurance company.

Reserves Process, $X = (X_t)_{t \geq 0}$: Available resources by the insurance company. Claims are modeled with negative jumps.

Dividend Process, $D = (D_t)_{t \geq 0}$: The company has to decide a dividend payment strategy $D$, which represents the cumulative payments. Adapted, non-negative, non-decreasing, càglàd processes with $D_0 = 0$.

Surplus Process, $X^D := X - D$: Process $D$ cannot lead to ruin, so $D_{t+} - D_t \leq X_t^D \lor 0$. Let $\Theta$ be the set of such strategies.
Objective Function: The company’s objective is to maximize

$$\nu^D(x_0) := \mathbb{E}_{x_0} \left[ \int_0^{\tau^D} e^{-qt} dD_t \right]$$
Objective Function: The company’s objective is to maximize

\[ \nu^D(x_0) := \mathbb{E}_{x_0} \left[ \int_0^{\tau^D} e^{-qt} dD_t \right] \]

- The flow of discounted dividend payments.
Objective Function: The company’s objective is to maximize

\[ \mathcal{V}^D(x_0) := \mathbb{E}_{x_0} \left[ \int_0^{\tau^D} e^{-qt} dD_t \right] \]

- The flow of discounted dividend payments.
- Time of ruin is given by \( \tau^D := \inf \{ t : X_t^D < 0 \} \).
Objective Function: The company’s objective is to maximize

$$\mathcal{V}^D(x_0) := \mathbb{E}_{x_0} \left[ \int_0^{\tau^D} e^{-qt} dD_t \right]$$

- The flow of discounted dividend payments.
- Time of ruin is given by $\tau^D := \inf\{ t : X^D_t < 0 \}$.

De Finetti’s problem [De Finetti, 1957] consists in finding

$$\sup_{D \in \Theta} \mathcal{V}^D(x_0).$$

The solution to this problem has been extensively studied for different classes of reserve processes ([Schmidli, 2008], [Loeffen, 2008].)
De Finetti’s Problem

De Finetti’s problem under a time of ruin constraint
  Problem formulation
  The Dual Problem

Exponential claims case

Spectrally Negative Lévy reserves
  Solution of dual problem
  Solution of the problem

Numerical example

Extensions
Motivation

There exists a trade-off between stability and profitability. The alternative problem of minimizing ruin probability means no dividend payment and profits are 0. On the contrary, maximizing the dividends leads to a dividend payment trend for which ruin is certain regardless of the initial amount $x_0$. 

Some problems have been proposed to deal with this trade-off: [Paulsen, 2003], [Thonhauser and Albrecher, 2007], [Loeffen and Renaud, 2010].
Motivation

There exists a trade-off between stability and profitability. The alternative problem of minimizing ruin probability means no dividend payment and profits are 0. On the contrary, maximizing the dividends leads to a dividend payment trend for which ruin is certain regardless of the initial amount $x_0$.

Some problems have been proposed to deal with this trade-off: [Paulsen, 2003], [Thonhauser and Albrecher, 2007], [Loeffen and Renaud, 2010].
The Problem

Inspired by previous work of [Thonhauser and Albrecher, 2007] we introduce a constraint on the time of ruin and state the following problem:

\[
V(x) := \sup_{D \in \Theta} \mathcal{V}^D(x)
\]

\[
\text{s.t. } \mathbb{E}_x \left[ \int_0^{\tau^D} e^{-qs} ds \right] \geq \int_0^T e^{-qs} ds \quad T \text{ fixed},
\]

Define \( K_T := \int_0^T e^{-qs} ds \).
In order to solve this problem we use Lagrangian relaxation to reformulate our problem. We first define the following function

\[
V^D_\Lambda(x) := \mathbb{E}_x \left[ \int_0^{\tau^D} e^{-qt} dD_t + \Lambda \int_0^{\tau^D} e^{-qs} ds \right] - \Lambda K_T
\]

for \( \Lambda \geq 0 \).

Then, (P) is equivalent to \( \sup \inf_{D \in \Theta} V^D_\Lambda(x) \) since

\[
\inf_{\Lambda \geq 0} V^D_\Lambda(x) = \begin{cases} 
V^D(x) & \text{if } \mathbb{E}_x \left[ \int_0^{\tau^D} e^{-\delta s} ds \right] \geq K_T \\
-\infty & \text{otherwise}.
\end{cases}
\]
The dual problem of (P) is defined as

$$\inf_{\Lambda \geq 0} \sup_{D \in \Theta} \mathcal{V}_\Lambda^D(x).$$  \hspace{1cm} (D)

Remember that \((P) \leq (D)\) always holds. Therefore, once (D) is solved, the main goal is to prove that there is no duality gap, that is

$$\sup_{D \in \Theta} \inf_{\Lambda \geq 0} \mathcal{V}_\Lambda^D(x) = \inf_{\Lambda \geq 0} \sup_{D \in \Theta} \mathcal{V}_\Lambda^D(x).$$
Recall

\[ \mathcal{V}_\Lambda^D(x) = \mathbb{E}_x \left[ \int_0^{\tau^D} e^{-qt} dD_t + \Lambda \int_0^{\tau^D} e^{-qt} dt \right] - \Lambda K_T. \]

To solve (D), we can use previous results on de Finetti’s problem and compute

\[ V_\Lambda(x) := \sup_{D \in \Theta} \mathcal{V}_\Lambda^D(x) \quad \text{(Pa)} \]
Recall

\[ V^D_\Lambda(x) = \mathbb{E}_x \left[ \int_0^{\tau^D} e^{-qt} dD_t + \Lambda \int_0^{\tau^D} e^{-qt} dt \right] - \Lambda K_T. \]

To solve (D), we can use previous results on de Finetti’s problem and compute

\[ V_\Lambda(x) := \sup_{D \in \Theta} V^D_\Lambda(x) \quad \text{(Pa)} \]

Where to start? A simple scenario.
De Finetti’s Problem

De Finetti’s problem under a time of ruin constraint
  Problem formulation
  The Dual Problem

Exponential claims case

Spectrally Negative Lévy reserves
  Solution of dual problem
  Solution of the problem

Numerical example

Extensions
Assume the reserves process follow the Cramér-Lundberg model with exponentially distributed claim sizes.

**Theorem**

Let \( x \geq 0 \) and \( V(x), V_\Lambda(x) \) be the optimal solutions to \((P)\), \((Pa)\), respectively. Then, strong duality holds, i.e.,

\[
\inf_{\Lambda \geq 0} V_\Lambda(x) = V(x).
\]

- Key point: **Barrier strategies are optimal.**

[Hernández and Junca, 2015]
Can we do more?

So far, we succeeded to solve (P) under the particular case of reserves that follow the Cramér-Lundberg model with claims that are exponentially distributed.
Can we do more?

So far, we succeeded to solve (P) under the particular case of reserves that follow the Cramér-Lundberg model with claims that are exponentially distributed.

But how about more general reserves processes?
De Finetti’s Problem

De Finetti’s problem under a time of ruin constraint
  Problem formulation
  The Dual Problem

Exponential claims case

**Spectrally Negative Lévy reserves**
  Solution of dual problem
  Solution of the problem

Numerical example

Extensions
Some definitions

- $X$ is a *spectrally negative Lévy process, SNLP*, if its Lévy triple $(a, \sigma, \Pi)$ is such that $\Pi$ is supported on $(-\infty, 0)$, i.e., they have no positive jumps, and do not have monotone paths a.s.
Some definitions

- $X$ is a *spectrally negative Lévy process*, SNLP, if its Lévy triple $(a, \sigma, \Pi)$ is such that $\Pi$ is supported on $(-\infty, 0)$, i.e., they have no positive jumps, and do not have monotone paths a.s.

- For SNLP we define the *Laplace exponent* and its inverse by

$$\psi(\theta) := \log(\mathbb{E}[e^{\theta X_1}]) \quad \theta \geq 0,$$

$$\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad q \geq 0.$$
Some definitions

- $X$ is a *spectrally negative Lévy process*, SNLP, if its Lévy triple $(a, \sigma, \Pi)$ is such that $\Pi$ is supported on $(-\infty, 0)$, i.e., they have no positive jumps, and do not have monotone paths a.s.

- For SNLP we define the *Laplace exponent* and its inverse by

  \[
  \psi(\theta) := \log(\mathbb{E}[e^{\theta X_1}]) \quad \theta \geq 0,
  \]

  \[
  \Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad q \geq 0.
  \]

- There exist a family of functions called $q$-scale $W^{(q)} : \mathbb{R} \to [0, \infty)$ defined for each $q \geq 0$ s.t. $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is a strictly increasing and continuous function on $[0, \infty)$ whose Laplace transform satisfies,

  \[
  \int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q} \quad \text{for } \beta > \Phi(q). \tag{1}
  \]
Some definitions

- $X$ is a *spectrally negative Lévy process*, SNLP, if its Lévy triple $(a, \sigma, \Pi)$ is such that $\Pi$ is supported on $(-\infty, 0)$, i.e., they have no positive jumps, and do not have monotone paths a.s.

- For SNLP we define the *Laplace exponent* and its inverse by

$$
\psi(\theta) := \log(\mathbb{E}[e^{\theta X_1}]) \quad \theta \geq 0,
$$

$$
\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad q \geq 0.
$$

- There exist a family of functions called $q$-scale $W^{(q)} : \mathbb{R} \to [0, \infty)$ defined for each $q \geq 0$ s.t. $W^{(q)}(x) = 0$ for $x < 0$ and $W^{(q)}$ is a strictly increasing and continuous function on $[0, \infty)$ whose Laplace transform satisfies,

$$
\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q} \quad \text{for } \beta > \Phi(q). \quad (1)
$$

- For each $q \geq 0$ define $Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(z)dz$, for $x \in \mathbb{R}$.
Consider a barrier strategy at level $b$, i.e., $D_t^b := (b \vee \bar{X}_t) - b$, $t \geq 0$, so $X_t^{D^b} = b - [(b \vee \bar{X}_t) - X_t]$. This motivates the following:

For a Lévy process $Y$ the reflected process at its supremum with initial value $s$ is defined as

$$
\hat{Y}_t^s := (s \vee Y_t) - Y_t, \quad t \geq 0.
$$

(2)

For such processes define also the exit time

$$
\hat{\sigma}^s_k := \inf \{ t > 0 : \hat{Y}_t^s > k \}.
$$

(3)

Therefore,

$$
X_t^{D^b} = b - \hat{X}_t^b
$$

and

$$
\tau^b := \tau^{D^b} = \hat{\sigma}^b.
$$
Figure: Process $X_t$ and process $(b \lor \bar{X}_t)$ (green line)

Figure: $X_t^{D^b} = b - \hat{X}_t^b$
The following results are the link between (D) and the $q$-scale functions:

**Theorem ([Gerber, 1972])**

Let $b > 0$ and consider the process $D^b$ as before. For $x \in [0, b]$,

$$
\mathbb{E}_x \left[ \int_0^{\tau^b} e^{-qs} dD_s^b \right] = \frac{W^{(q)}(x)}{W^{(q)'}(b+)},
$$

where $W^{(q)'}(b+)$ is understood as the right derivative of $W^{(q)}$ at $b$. 

Theorem ([Avram et al., 2004])

Let \( b > 0 \) and consider the process \( D^b \) as before. For \( x \in [0, b] \),

\[
\mathbb{E}_x \left[ e^{-q\tau^b} \right] = Z(q)(x) - q \frac{W(q)(x)}{W(q)'(b)} W(q)(b).
\]

Therefore,

\[
\mathbb{E}_x \left[ \int_0^{\tau^b} e^{-qs} \, ds \right] = \frac{W(q)(x)}{W(q)'(b)} W(q)(b) - \int_0^x W(q)(z) \, dz. \tag{5}
\]
Expression (5) and (4) yields the following result.

**Proposition ([Loeffen, 2009])**

For a sufficiently smooth q-scale function $W^{(q)}$, the value of (L) for the barrier strategy at level $b \geq 0$ is given by

$$V^D_b(x) = \begin{cases} W^{(q)}(x) \left[ 1 + \Lambda W^{(q)}(b) \right] - \Lambda \int_0^x W^{(q)}(z) dz - \Lambda K_T & \text{if } x \leq b \\ x - b + V^D_b(b) & \text{if } x > b. \end{cases}$$

(6)
Expression (5) and (4) yields the following result.

**Proposition ([Loeffen, 2009])**

For a sufficiently smooth q-scale function $W(q)$, the value of (L) for the barrier strategy at level $b \geq 0$ is given by

$$V_{\Lambda}^{D^b}(x) = \begin{cases} W(q)(x) \left[ \frac{1+\Lambda W(q)(b)}{W(q)'(b)} \right] - \Lambda \int_0^x W(q)(z)dz - \Lambda K_T & \text{if } x \leq b \\ x - b + V_{\Lambda}^{D^b}(b) & \text{if } x > b. \end{cases}$$

(6)
Let $\zeta_{\Lambda} : [0, \infty) \to \mathbb{R}$ be defined by

$$\zeta_{\Lambda}(x) := \frac{1 + \Lambda W^{(q)}(x)}{W^{(q)'}(x)}, \quad x > 0$$

and $\zeta_{\Lambda}(0) := \lim_{x \downarrow 0} \zeta_{\Lambda}(x)$. In [Loeffen, 2009] it is shown that the barrier strategy at level

$$b^*_\Lambda := \sup\{b : \zeta_{\Lambda}(b) \geq \zeta_{\Lambda}(x), \text{ for all } x \geq 0\}$$

is the optimal strategy for (Pa). Furthermore, when the Lévy measure $\Pi$ has completely monotone density, the function $\zeta_{\Lambda}$ has a unique maximum.
Completely monotone functions

Definition
A function $f$ is said to be completely monotone if $f \in C[0, \infty)$, $f \in C^\infty(0, \infty)$ and satisfies $(-1)^n \frac{d^n}{dx^n} f(x) \geq 0$

Example

$$\frac{1}{(\lambda + \mu x)^\nu}, \quad \ln \left( b + \frac{c}{x} \right), \quad e^{\frac{a}{x}}, \quad \frac{\ln(1 + x)}{x},$$

where $\mu, \lambda, \nu \geq 0$ with no both $\lambda$ and $\mu$ equal to zero and $b \geq 1, a, c > 0$. 
Value Function of (Pa)

Theorem ([Loeffen, 2009])

Suppose the Lévy measure of the spectrally negative Lévy process $X$ with Lévy triple $(\gamma, \sigma, \Pi)$, has a completely monotone density. Let $c = \gamma + \int_0^1 x\Pi(dx)$. Then the following holds:

(i) If $\sigma = 0$ and $\Pi(0, \infty) < \infty$ and $\Lambda < -c/q$ the take-the-money-and-run strategy is an optimal strategy for (Pa).

(ii) if $\sigma > 0$, or $\Pi(0, \infty) = \infty$, or $\Pi(0, \infty) < \infty$ and $\Lambda \geq -c/q$, then the optimal strategy consists of a barrier strategy at level $b^*_\Lambda$ given by equation (8) and the corresponding value function is given by equation (6).
Proposition

For each $b \in (b_0^*, \infty)$ there exists a $\Lambda$ such that, $D^b$, the barrier strategy at level $b$, is optimal for $(Pa)$ with $\Lambda$.

Theorem

Let $x \geq 0$, $K_T \geq 0$, and $V(x)$, $V_\Lambda(x)$ be the optimal solution to (P), (Pa), respectively, with initial value $x$, then, strong duality holds, i.e.,

$$\inf_{\Lambda \geq 0} V_\Lambda(x) = V(x).$$
Proof. Let $\tau_b$ be the time of ruin of the barrier strategy with level $b$ and initial value $x \in [0, b]$. Let

$$\Psi_b(x) = \mathbb{E}_x \left[ \int_0^{\tau_b} e^{-qs} ds \right].$$

To obtain the optimal value function of (P) we need to show $(b^*, \Lambda^*)$ that certify strong duality. To do so we distinguish the following 4 cases depending on the initial value $x$ and the parameter $T$: 
Figure: Characterization of the solution of (P).
Numerical Example

We illustrate the previous result for $X$ with Lévy triple $(10, 0.1, \frac{d}{|x|^{1.8}} 1_{(-\infty, 0)})$, that is an 0.8–stable process with diffusion.
Figure: $\Psi_b(x)$ for different barrier levels. Inactive and biding constraint cases.
Figure: $V_\Lambda(x)$ for different initial values. Inactive and bidding constraint cases.
Figure: Value functions for constrained and unconstrained problems for different values of $K_T$. 
De Finetti’s Problem

De Finetti’s problem under a time of ruin constraint
  Problem formulation
  The Dual Problem

Exponential claims case

Spectrally Negative Lévy reserves
  Solution of dual problem
  Solution of the problem

Numerical example

Extensions
Extensions

- Dual model (spectrally positive Levy Processes).
- Optimal dividend with fixed transaction cost. (Impulse control and \((S, s)\) strategy).
- Study other types of restrictions such that:

\[
\mathbb{E}_x[\tau^D] \geq T
\]

\[
\mathbb{P}(\tau^D \leq T | X_0^D = x_0) \leq \epsilon
\]


De Finetti's optimal dividends problem with an affine penalty function at ruin.

Optimal dividend payouts for diffusions with solvency constraints.

*Stochastic Control in Insurance.*
Probability and Its Applications. Springer.

Dividend maximization under consideration of the time value of ruin.
Thank You