AN OPTIMAL DIVIDEND PROBLEM

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March 31, 2016

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PRELIMINARIES
Goal: Find the value of a cash flow under a liquidity constraint.
Cash flow rate:

\[ \mu_t = \nu + \int_0^t k(\bar{\mu} - \mu_s) ds + \sigma \tilde{W}_t \]
Cash reserves dynamics:

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Cumulative dividends until time \( t \):

\[ L_t \quad \text{RCLL, non-decreasing.} \]
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Cumulative dividends until time \( t \):
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Cash reserves at time \( t \):
\[ X_t = x + \int_0^t \mu_s ds + \sigma W_t - L_t \]
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- Deterministic: \( \sigma = \bar{\sigma} = 0 \).
- Semi-deterministic: \( \sigma = 0 \) or \( \bar{\sigma} = 0 \).
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\[ V(x, \nu) = \sup_L E \left[ \int_0^{\theta(L)} e^{-rt} dL_t \right], \]

where \( \theta(L) \) is the time the cash reserves reach 0 under the dividend policy \( L \) (bankruptcy time).
DETERMINISTIC CASE
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Since everything is explicit, we find that

$$\mu_t = \bar{\mu} + (\nu - \bar{\mu}) e^{-kt}$$

and

$$V(x, \nu) = x + \frac{\bar{\mu}}{r} + \frac{\nu - \bar{\mu}}{r + k}, \quad \nu \geq 0.$$
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If $x$ is lower, bankruptcy will come before positive cash flows, otherwise we can wait for better times.

$$V(x, \nu) = x + \max \left\{ 0, e^{-r\tau_0(\nu)} \frac{k\bar{\mu}}{r(r + k)} - x_b(\nu) \right\}.$$
Below $\mu_{\text{min}}$ voluntary liquidation is optimal at all cash levels.
GENERALLY...
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$$V(x, \nu) \equiv x, \quad \text{for } \nu \leq \mu^*.$$
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Cute proof: First observe that \[ V(x, \nu) \leq x + \sup_{\tau} E \left[ \int_0^\tau e^{-rt} \mu_t \, dt \right] = x + V^R(\nu). \]

Then create supersolution $V^R$ which attains zero.
What we know:

• Optimal policy exists.
• The value function is continuous.
• Dynamic programming holds.
• The dynamic programming equation has the comparison property ($\approx$ unique solution).
Pay dividends
No dividends
THANK YOU!