AN OPTIMAL DIVIDEND PROBLEM

Max Reppen (joint work with Mete Soner and Jean-Charles Rochet)

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ETH Zürich

PRELIMINARIES

Goal: Find the value of a cash flow under a liquidity constraint.

CASH RESERVES DYNAMICS

Cash flow rate:

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- Deterministic: $\sigma = \tilde{\sigma} = 0$.
- Semi-deterministic: $\sigma = 0$ or $\tilde{\sigma} = 0$.

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$$V(x,\nu) = \sup_L E\left[\int_0^{\theta(L)} e^{-rt} dL_t\right],$$

where $\theta(L)$ is the time the cash reserves reach 0 under the dividend policy L (bankruptcy time).

DETERMINISTIC CASE

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Since everything is explicit, we find that

$$\mu_t = \bar{\mu} + (\nu - \bar{\mu})e^{-kt}$$

and

$$V(x,\nu) = x + \frac{\bar{\mu}}{r} + \frac{\nu - \bar{\mu}}{r+k}, \quad \nu \geq 0.$$

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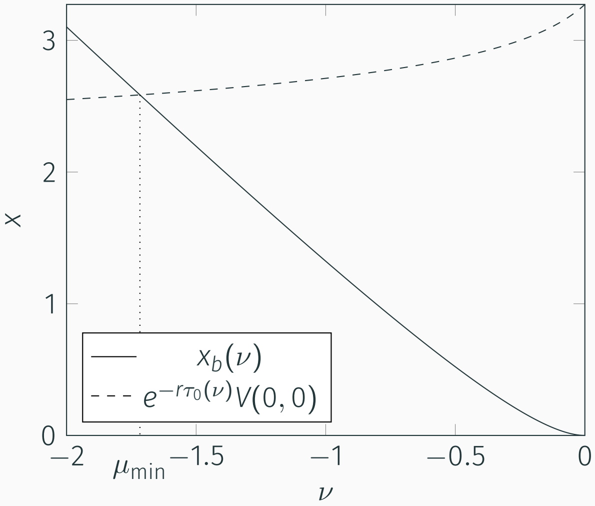
If x is lower, bankruptcy will come before positive cash flows, otherwise we can wait for better times.

$$V(x,\nu) = x + \max\left\{0, e^{-r\tau_0(\nu)} \underbrace{\frac{k\bar{\mu}}{r(r+k)}}_{V(0,0)} - x_b(\nu)\right\}.$$



Below $\mu_{\min}$ voluntary liquidation is optimal **at all cash levels**.

GENERALLY…

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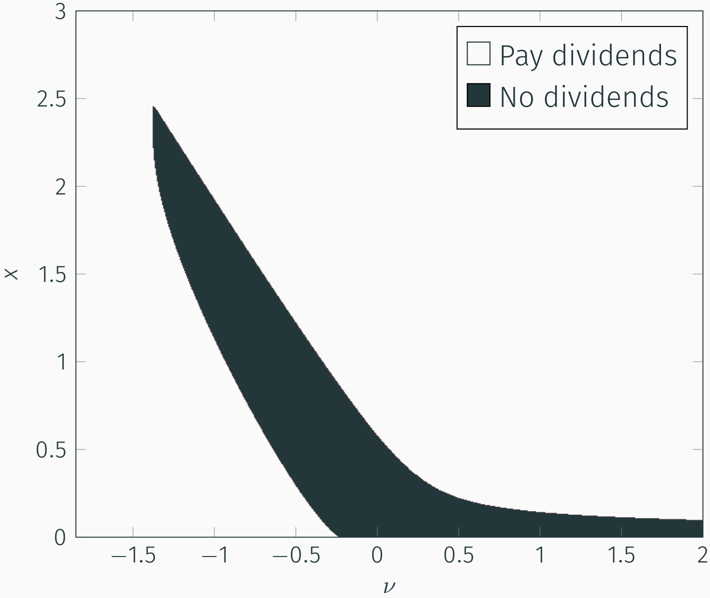
Cute proof: First observe that

$$V(x,\nu) \le x + \sup_\tau E\left[\int_0^\tau e^{-rt}\mu_t dt\right] = x + V^R(\nu).$$

Then create supersolution V^R which attains zero.

What we know:

- Optimal policy exists.
- The value function is continuous.
- Dynamic programming holds.
- The dynamic programming equation has the comparison property ($\approx$ unique solution).



Pay dividends
No dividends
3
2.5
2
1.5
1
0.5
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x
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−1
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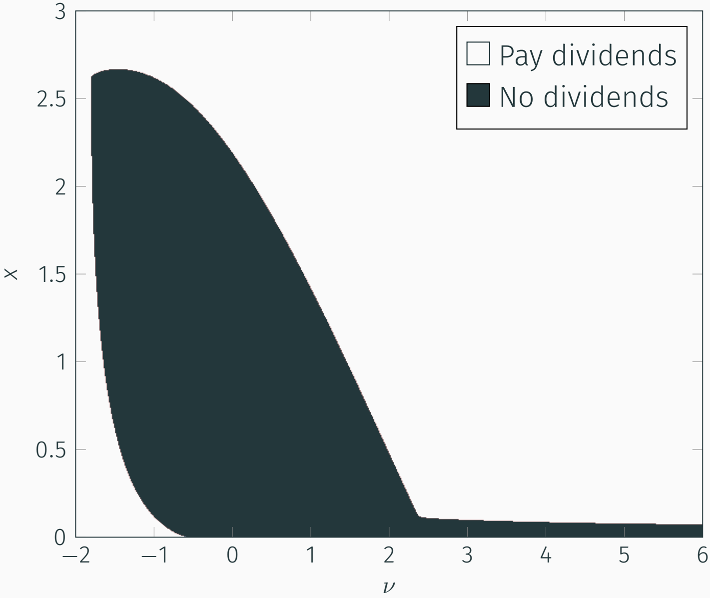
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Pay dividends
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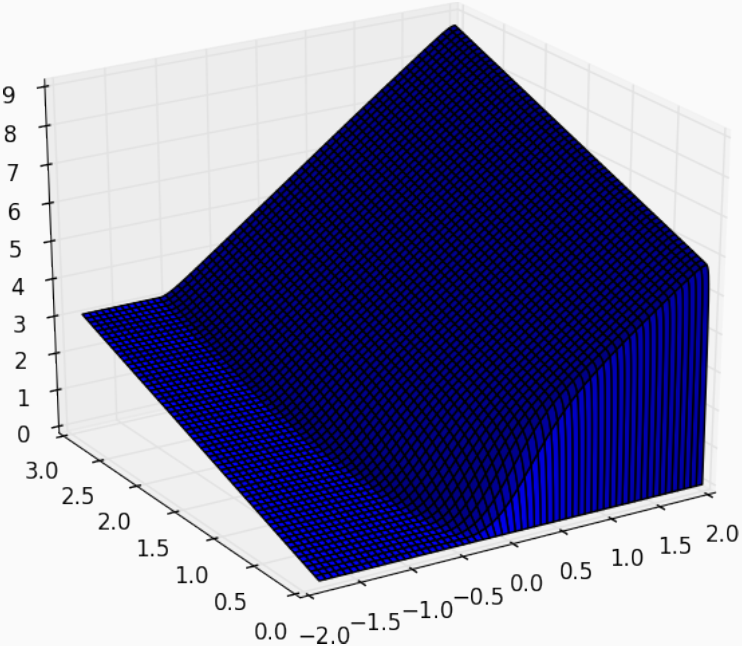
9
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THANK YOU!