Hamilton-Jacobi approach for some stochastic control problems with state constraints

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Outline

1. Introduction
2. Swing options in energy market
3. State-constrained stochastic problems: general setting
   - Step 1: reachability problem using unbounded controls
   - Step 2: level set approach
   - Step 3: HJB equation and uniqueness result

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Introduction - Motivation

Stochastic differential equations. State and control variables

Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Consider the following controlled SDE in $\mathbb{R}^d$:

$$\begin{cases} dX(s) = b(X(s), u(s))ds + \sigma(X(s), u(s))dB_s \quad s \in [t, T] \\ X(t) = x \end{cases} \tag{1}$$

where $B.$ is a Brownian motion and

$$u \in \mathcal{U} := \{\text{Progr. meas. processes with values in } U\},$$

for a given <u>compact set U</u>.

$\rightsquigarrow$ $\exists\ X(\cdot) := X^u_{t,x}(\cdot)$: unique strong solution of (1) associated to the control u (under standard regularity assumptions on b,σ).

Introduction - Setting of the problem

State-constrained OCPs

- Let $\psi : \mathbb{R}^d \to \mathbb{R}$ (terminal cost) and $\ell : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ (running cost).
- Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a non empty and closed set.

STATE-CONSTRAINED OCP:

$$\vartheta(t,x) = \inf_{u(\cdot)\in\mathcal{U}} \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(X^u_{t,x}(s), u(s))\, ds \right] \text{ s. t. } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t, T] \text{ a.s.} \right\}.$$

A particular case: No control,

$$\vartheta(t,x) = \begin{cases} \mathbb{E}\left[\psi(X_{t,x}(T))\right] & \text{if } X_{t,x}(\cdot) \subset \mathcal{K} \text{ a.s.,} \\ +\infty & \text{otherwise.} \end{cases}$$

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Deterministic case: without state constraints

$$\vartheta(t,x) = \inf_{u\in\mathcal{U}} \psi(X^u_{t,x}(T)) + \int_t^T \ell(X^u_{t,x}(s), u(s))\, ds$$

- Dynamic Programming Principle (DPP): for $0 < h \leq T - t$

$$\vartheta(t,x) = \inf_{u\in\mathcal{U}} \left[\vartheta(t+h, X^u_{t,x}(t+h)) + \int_t^{t+h} \ell(X^u_{t,x}(s), u(s))\, ds \right].$$

- The HJB equation associated to ϑ is:

$$-\partial_t \vartheta(t,x) + H(x, D\vartheta) = 0 \quad \text{on } (0,T)\times\mathbb{R}^d,$$
$$\vartheta(T,x) = \psi(x).$$

with $H(x,p) := \sup_{u\in U}(-b(x,u)\cdot p - \ell(x,u))$.

- The HJB equation describes a "Front Propagation" phenomena.

Deterministic case: without state constraints

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- The HJB equation describes a "Front Propagation" phenomena.

Stochastic case: without state constraints

$$\vartheta(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(X^u(s),u(s))\,ds\big]$$

- The DPP reads as (for $0 < h \leq T - t$)

$$\vartheta(t,x) = \inf_{u\in\mathcal{U}} \mathbb{E}\big[\vartheta(t+h, X^u_{t,x}(t+h)) + \int_t^{t+h} \ell(X^u_{t,x}(s),u(s))\,ds\big]$$

- and the HJB equation is

$$-\partial_t\vartheta + H(x, D\vartheta, D^2\vartheta)) = 0$$
$$\vartheta(T,x) = \psi(x)$$

with $H(x,p,Q) := \sup_{u\in U} \big(- b(x,u)\cdot p - \ell(x,u) - \frac{1}{2} Tr(\sigma\sigma^T Q)\big).$

Viscosity Notion

- The notion of viscosity solution can be used to give a rigorous sense to the solutions of HJB equations
- In many cases the value function is merely continuous or Lipschitz continuous (no more)
- The viscosity notion provides a convenient framework for numerical analysis and error estimates theory. For a consistent, stable and monotone scheme, we get: $\|\vartheta - \vartheta^h\| = 0(h^{1/2})$.
- Once the value function ϑ is known, the optimal control strategy can be computed as:

$$u^*(t,x) \in \text{argmax} H(x, D\vartheta(t,x), D^2\vartheta(t,x)).$$

Stochastic case: A simple example

- Consider the uncontrolled case (dim=1):

$$\begin{cases} dX(s) = b(X(s))ds + \sigma(X(s))dB_s \quad s \in [t, T] \\ X(t) = x \end{cases}$$

- Define the value function by:

$$\vartheta_{BS}(t, x) = \mathbb{E}\big[\psi(X_{t,x}(T))\big].$$

- The HJB equation associated to ϑ (Black & Scholes eq.):

$$-\partial_t \vartheta_{BS} - b(x) \cdot D\vartheta_{BS} - \frac{\sigma(x)^2}{2} \Delta \vartheta_{BS} = 0$$
$$\vartheta_{BS}(T, x) = \psi(x)$$

In the sequel, we would like to describe the value function ϑ in presence of constraints and obstacles.

Applications

- aircraft landing problem with wind disturbances
- energy consumption/demand problem with limited resource
- super replication problems in finance
- ... any stochastic problem with state constraints

Characterize ϑ and design numerical methods for its approximation

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Swing options in energy market

- Assume $Y_{t,y}(\cdot)$ is the "price of energy" satisfying:

$$dY_{t,y}(s) = b(Y_{t,y}(s))ds + \sigma(Y_{t,y}(s))dB_s, \qquad Y_{t,y}(t) = y$$

- Let $T > 0$ be the time maturity of the contract, and K is a pre-specified strike price.

- At every $s \in [0, T]$, possibility to buy an amount $u(s) \in [0, \bar{u}]$ at price K.

- The cost functional to be maximized:

$$J(Y_{t,y}) = \int_t^T u(s)(Y_{t,x}(s) - K)\, ds.$$

- Limited ressources:

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- Limited ressources:

$$0 \leq \zeta_{t,z} := z + \int_t^T u(s)\, ds \leq M.$$

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- The cost functional to be maximized:

$$J(Y_{t,y}) = \int_t^T u(s)(Y_{t,x}(s) - K)\, ds.$$

- Limited ressources:

$$d\zeta_{t,z}(s) = u(s)ds, \quad \zeta_{t,z}(t) = z,$$

$$0 \le \zeta_{t,z}(s) \le M \quad \forall s \in [t, T].$$

Swing options in energy market

- The state $(Y_{t,y}(\cdot), \zeta(\cdot))$ satisfies:

$$\begin{array}{rcll} dY_{t,y}(s) & = & b(Y_{t,y}(s)) + \sigma(Y_{t,y}(s))dB_s, & Y_{t,y}(t) = y \\ d\zeta_{t,z}(s) & = & u(s), & \zeta_{t,z}(t) = z. \end{array}$$

- The control variable: $u(s)) \in [0, \bar{u}]$.
- State constraints: $0 \leq \zeta_{t,z}(s) \leq M$ for every $s \in [t, T]$.
- Formulation of the control problem (for $x = (y, z)$): .

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- Formulation of the control problem (for $x = (y, z)$):

$$\vartheta(t, x) = \min_{u \in \mathcal{U}} \{ J(Y_{t,y}(T)), \;\; 0 \leq \zeta_{t,z}(T) \leq M \}$$

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- The control variable: $u(s)) \in [0, \bar{u}]$.
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- Formulation of the control problem (for $x = (y, z)$):

$$\vartheta(t,x) = \min_{u \in \mathcal{U}} \{ J(Y_{t,y}(T)), \quad m(s) \leq \zeta_{t,z}(s) \leq M(s) \}.$$

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Swing options in energy market: some references.

- Discrete time:

  - O. Bardou, S. Bouthemy, G. Pages. *Optimal quantization for the pricing of swing options*. Appl. Math. Finance, 2009

  - C. Barrera-Esteve, F. Bergeret, C. Dossal, E. Gobet, A. Meziou, R.Munos, D. Reboul-Salze. *Numerical methods for the pricing of swing options: a stochastic control approach*. Methodol. Comput. Appl. Probab, 2006.

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  - ...

- Continuous time

  - R. Carmona, N. Touzi. *Optimal multiple stopping and valuation of swing options*. Math. Finance, 2008.

  - F. E. Benth, J. Lempa, T. K. Nilssen. *On the optimal exercise of swing options in electricity markets*. The Journal of Energy Markets, 2012.

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Introduction

State-constrained OCPs

$$\vartheta(t,x) = \min_{u\in\mathcal{U}} \mathbb{E}\left[\psi(X^u_{t,x}(T))\right] \text{ on processes s.t. } X^u_{t,x}(\cdot) \subset \mathcal{K},$$

with

$$\begin{cases} dX(s) = b(s,X(s),u(s))ds + \sigma(s,X(s),u(s))dB_s & s\in[t,T] \\ X(t) = x \end{cases}$$

- Unconstrained case ($\mathcal{K} = \mathbb{R}^d$): ϑ is the unique continuous viscosity solution of a second order HJB equation;
- Constrained case: Non uniqueness issues due to the loss of regularity of the value function.
  - Deterministic case: Soner ('86), Blanc ('97), Frankowska-Vinter ('00), Hermosilla-HZ ('15), Altarovici-Bokanowski-HZ ('13) ....
  - Stochastic case: Lasry-Lions ('89), Katsoulakis ('94), Bouchard-Nutz ('12) ....

Stochastic case

Stochastic OCP with state constraints

- Characterisation of viability kernels:
  Aubin- Da Prato (90,98), ...
  Buckdahn, Peng, Quincampoix, Rainer (98)
  Quincampoix-Rainer (05)
- Stochastic target problems
  Soner-Touzi (00), (02), (09)
  Bouchard-Elie-Imbert (10)

$\Rightarrow$ **A strong compatibility condition is required**

- $\forall x, \forall p \neq 0, \exists a \in U, \langle \sigma(x,a), p \rangle = 0$
- invertibility condition on the matrix σ
- or other qualification conditions ...

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State-constrained stochastic problems: general setting

Consider the following controlled SDE in $\mathbb{R}^d$:

$$\begin{cases} dX(s) = b(X(s), u(s))ds + \sigma(X(s), u(s))dB_s & s \in [t, T] \\ X(t) = x \end{cases} \tag{2}$$

where

$$u \in \mathcal{U} := \{\text{Progr. meas. processes with values in } U\}.$$

The STATE-CONSTRAINED OCP is:

$$\vartheta(t,x) = \inf_{u \in \mathcal{U}} \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T))\right] \mid X^u_{t,x}(\cdot) \subset \mathcal{K} \text{ a.s.} \right\}$$

State-constrained stochastic problems: general setting

Assumptions:

(S1) $u(\cdot) \in U$ a.s., with $U \subseteq \mathbb{R}^m$ compact set;

(S2) $b, \sigma : \mathbb{R}^d \times U \to \mathbb{R}^d, \mathbb{R}^{d\times p}$ are continuous functions, Lipschitz continuous w.r.t. x (unif. in u).

$\rightsquigarrow$ $X^u_{t,x}(\cdot)$ unique strong solution of (2).

(S3) $\psi : \mathbb{R}^d \to \mathbb{R}$ Lipschitz continuous.

(S4) $\psi \geq 0$

State-constrained stochastic problems: general setting

Step 1: reachability problem using unbounded controls

$$\vartheta(t,x) \quad = \quad \inf_{u\in\mathcal{U}} \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T)) \right] \mid X^u_{t,x}(\cdot) \subset \mathcal{K} \text{ a.s.} \right\}$$

It is straightforward to obtain :

$$\vartheta(t,x) = \quad \inf \Big\{ z \in \mathbb{R} \ \mid \ \exists u,\ \mathbb{E}\left[\psi(X^u_{t,x}(T)) \right] \leq z \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \Big\}$$

State-constrained stochastic problems: general setting

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State-constrained stochastic problems: general setting

Step 1: reachability problem using unbounded controls

Thanks to Ito's representation theorem this is OK up to a martingale: $\exists \alpha \in \mathcal{A} := L^2_{\mathbb{F}}$ ($\mathbb{R}^p$-valued prog. meas. process):

$$\psi(X^u_{t,x}(T)) = \mathbb{E}\big[\psi(X^u_{t,x}(T))\big] + \int_t^T \alpha_s \cdot dB_s, \quad \text{a.s.}$$

$$z \geq \mathbb{E}\Big[\psi(X^u_{t,x}(T))\Big]$$

$$\iff \quad \exists \alpha \in \mathcal{A},\ z \geq \psi(X^u_{t,x}(T)) - \int_t^T \alpha_s \cdot dB_s, \quad \text{a.s.}$$

State-constrained stochastic problems: general setting

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$$z \geq \mathbb{E}\Big[\psi(X^u_{t,x}(T))\Big]$$

$$\iff \quad \exists \alpha \in \mathcal{A},\ z \geq \psi(X^u_{t,x}(T)) - \int_t^T \alpha_s \cdot dB_s, \quad \text{a.s.}$$

State-constrained stochastic problems: general setting

Step 1: reachability problem using unbounded controls

Let

$$Z^{u,\alpha}_{t,x,z}(\cdot) := z + \int_t^{\cdot} \alpha_s \cdot dB_s. \tag{3}$$

In particular,

$$z \geq \mathbb{E}\Big[\psi(X^u_{t,x}(T))\Big] \iff \exists \alpha \in \mathcal{A}, \;\; Z^{\alpha,u}_{t,x,z}(T) \geq \psi(X^u_{t,x}(T)) \quad \text{a.s.}$$
$$\iff \exists \alpha \in \mathcal{A}, \; (X^u_{t,x}(T), Z^{\alpha,u}_{t,x,z}(T)) \in \mathrm{epi}(\psi) \quad \text{a.s.}$$

State-constrained stochastic problems: general setting

Step 1: reachability problem using unbounded controls

⇒ We consider the following stochastic state-constrained backward reachable set

$$\mathcal{R}_t^{\psi,\mathcal{K}} := \Big\{(x,z) \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U} \times \mathcal{A} \text{ such that}$$
$$(X_{t,x}^u(T), Z_{t,x,z}^{u,\alpha}(T)) \in \text{epi}(\psi) \text{ and } X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.}\Big\}.$$

Then

$$\vartheta(t,x) = \inf\Big\{z \in \mathbb{R} : (x,z) \in \mathcal{R}_t^{\psi,\mathcal{K}}\Big\}.$$

State-constrained stochastic problems: general setting

Step 2: level set approach

Let

$$g_\psi(x,z) := \max(\psi(x) - z, 0) \quad \text{and} \quad g_\mathcal{K}(x) := d(x, \mathcal{K}).$$

In particular:

$$g_\psi, g_\mathcal{K} \geq 0 \quad \text{and} \quad g_\psi(x,z) = 0 \Leftrightarrow (x,z) \in \text{epi}(\psi), \quad g_\mathcal{K}(x) = 0 \Leftrightarrow x \in \mathcal{K}.$$

AUXILIARY UNCONSTRAINED OCP:

$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[\underbrace{g_\psi(X^u_{t,x}(T), Z^{u,\alpha}_{t,x,z}(T))}_{\equiv \max(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,x,z}(T),\ 0)} + \int_t^T \underbrace{g_\mathcal{K}(X^u_{t,x}(s))}_{\equiv d_\mathcal{K}(X^u_{t,x}(s))} ds \Big]$$

State-constrained stochastic problems: general setting

Step 2: level set approach

Proposition

Then

$$\mathcal{R}_t^{\psi,\mathcal{K}} = \left\{(x, z) \in \mathbb{R}^{d+1} : w(t, x, z) = 0\right\}$$

In particular,

$$\vartheta(t, x) = \inf \left\{ z \in \mathbb{R} : w(t, x, z) = 0 \right\}.$$

- But α_s is unbounded ...

Some References (unbounded controls)

- Lasry - Lions (Cras '00) : "A new class of sing. sto. cont. pbs." ($b(x, u) = b_1(x) + b_2(x)u$).
- Pham (Prob. Surveys '05) : HJB equation for unbounded controls
- Brüder (Preprint HAL '05) : comparison principle for a super-replication problem in Finance
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- Debrabant-Jakobsen (Math. Comp. '13) : high order SL schemes

State-constrained stochastic problems: general setting

Step 3: HJB equation and uniqueness result

Recall that

$$\begin{pmatrix} dX_s \\ dZ_s \end{pmatrix} = \begin{pmatrix} b(X_s, u(s)) \\ 0 \end{pmatrix} ds + \begin{pmatrix} \sigma(X_s, u(s)) \\ \alpha \end{pmatrix} dW_s$$

and

$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\max(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,x,z}(T),\ 0) + \int_t^T d_{\mathcal{K}}(X^u_{t,x}(s))\right]$$

The HJB equation associated to this AUXILIARY OCP would be, in the case $p = d = 1$:

$$\underbrace{-\partial_t w + \sup_{\substack{u\in U,\\ \alpha\in\mathbb{R}}}\left\{ - b\partial_x w - \frac{1}{2}\sigma^2\partial_{xx} w - \alpha\sigma\partial_{xz} w - \frac{1}{2}\alpha^2\partial_{zz} w - d_{\mathcal{K}}(x)\right\}}_{=:H(x,z,Dw,D^2w)} = 0$$

$\Rightarrow$ Because of unbounded controls, H can be unbounded !

State-constrained stochastic problems: general setting

Step 3: HJB equation and uniqueness result

Recall that

$$\begin{pmatrix} dX_s \\ dZ_s \end{pmatrix} = \begin{pmatrix} b(X_s, u(s)) \\ 0 \end{pmatrix} ds + \begin{pmatrix} \sigma(X_s, u(s)) \\ \alpha \end{pmatrix} dW_s$$

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$$w(t,x,z) = \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\max(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,x,z}(T),\ 0) + \int_t^T d_{\mathcal{K}}(X^u_{t,x}(s))\right]$$

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$$\underbrace{\sup_{\substack{u\in U \\ \alpha\in\mathbb{R}}}\left\{-\partial_t w + -b\partial_x w + \ell\partial_z w - \frac{1}{2}\sigma^2\partial_{xx} w - \alpha\sigma\partial_{xz} w - \frac{1}{2}\alpha^2\partial_{zz} w - d_{\mathcal{K}}(x)\right\}}_{=:\widetilde{H}(x,z,\partial_t w, Dw, D^2 w)} = 0$$

$\Rightarrow$ Because of unbounded controls, H can be unbounded !

State-constrained stochastic problems: general setting

Interpretation of the eq. $\widetilde{H}(x, z, \partial_t w, Dw, D^2 w) = 0$

Notice that (if $d = p = 1$)

$$\sup_{\alpha \in \mathbb{R}} \left(A - 2B\alpha + C\alpha^2 \right) = 0 \quad \Leftrightarrow \quad A \leq 0,\ AC \leq B^2$$

Furthermore,

$$\sup_{\alpha \in \mathbb{R}} A - 2B\alpha + C\alpha^2 = 0$$

$$\Leftrightarrow \quad \sup_{\alpha_1^2+\alpha_2^2=1,\ \alpha_1 \neq 0} A - 2B\frac{\alpha_2}{\alpha_1} + C(\frac{\alpha_2}{\alpha_1})^2 = 0$$

$$" \Leftrightarrow " \quad \sup_{\alpha_1^2+\alpha_2^2=1} \mathbf{A}\alpha_1^2 - \mathbf{2B}\alpha_1\alpha_2 + \mathbf{C}\alpha_2^2 = \mathbf{0}$$

$$\Leftrightarrow \quad \sup_{(\alpha_1,\alpha_2)\in S^1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}^T \begin{pmatrix} A & -B \\ -B & C \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

$$\Leftrightarrow \quad \mathbf{\Lambda}_+ \begin{pmatrix} \mathbf{A} & \mathbf{-B} \\ \mathbf{-B} & \mathbf{C} \end{pmatrix} = 0$$

State-constrained stochastic problems: general setting

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State-constrained stochastic problems: general setting

Interpretation of the eq. $\widetilde{H}(x, z, \partial_t w, Dw, D^2 w) = 0$

Define the matrix :

$$\mathcal{L}^u(x, z, \partial_t, Dw, D^2 w) := \begin{pmatrix} A & -B \\ -B & C \end{pmatrix}$$

where

$$\frac{A}{2} := -\partial_t w - b \cdot D_x w - \frac{1}{2} Tr(\sigma\sigma^T D_x^2 w) - d(x, \mathcal{K})$$

$$B := \sigma \partial_{xz} w, \quad \text{and} \quad C := -\partial_{zz} w$$

By elementary calculus

$$\widetilde{H}(x, z, w_t, w, D^2 w) = 0$$

$$\Leftrightarrow \sup_{\substack{u \in U \\ \alpha_1^2 + \alpha_2^2 = 1}} \left\{ \alpha_1^2 (-\partial_t w - b\partial_x w - \frac{1}{2}\sigma^2 \partial_{xx} w - d(x, \mathcal{K})) - \alpha_1 \alpha_2 \sigma \partial_{xz} w - \frac{1}{2} \alpha_2^2 \partial_{zz} w \right\} = 0$$

$$\Leftrightarrow \sup_{u \in U} \Lambda^+ \Big(\mathcal{L}^u(x, z, \partial_t w, Dw, D^2 w) \Big) = 0.$$

State-constrained stochastic problems: general setting

Interpretation of the eq. $\widetilde{H}(x, z, \partial_t w, Dw, D^2 w) = 0$, for $d \geq 1$, $p \geq 1$

Define the matrix :

$$\mathcal{L}^u(x, z, \partial_t w, Dw, D^2 w) := \begin{pmatrix} A & -B_1 & \dots & -B_p \\ -B_1 & C & & 0 \\ \vdots & & \ddots & \vdots \\ -B_p & 0 & \dots & C \end{pmatrix}$$

where

$$\frac{A}{2} := -\partial_t w - b \cdot D_x w - \frac{1}{2} Tr(\sigma \sigma^T D_x^2 w) - d(x, \mathcal{K}).$$

$$B := \sigma^T D_x \partial_z w = (B_1, \dots, B_p)^T, \quad \text{and} \quad C := -\partial_{zz} w$$

By elementary calculus:

$$\widetilde{H}(x, z, w_t, Dw, D^2 w) = 0$$

$$\Leftrightarrow \sup_{u \in U} \Lambda^+ \Big(\mathcal{L}^u(x, z, \partial_t w, Dw, D^2 w) \Big) = 0.$$

State-constrained stochastic problems: general setting

Step 3: HJB equation and uniqueness result

Theorem

The auxiliary value function w is a viscosity solution of the following generalized HJB equation

$$\begin{cases} \sup_{u\in U} \Lambda^+\Big(\mathcal{L}^u(x,z,Dw,D^2w)\Big) = 0, & t < T, (x,z) \in \mathbb{R}^{d+1} \\ w(T,x,z) = g_\psi(x,z) & \mathbb{R}^{d+1} \end{cases}$$

in the class of continuous function with linear growth at infinity $(|w(t,x,z)| \leq C(1+|x|+|z|))$

What about uniqueness ??

Uniqueness issue: the uncontrolled case.

Consider the solution to the HJB equation:

$$\boldsymbol{\Lambda}^+\begin{pmatrix} -\frac{\partial\vartheta}{\partial t}-b(t,x)\frac{\partial\vartheta}{\partial y}-\frac{1}{2}\sigma^2(t,x)\frac{\partial^2\vartheta}{\partial x^2} & -\frac{1}{2}\sigma(t,x)\frac{\partial^2\vartheta}{\partial x\partial z} \\ -\frac{1}{2}\sigma(t,x)\frac{\partial^2\vartheta}{\partial x\partial z} & -\frac{1}{2}\frac{\partial^2\vartheta}{\partial z^2} \end{pmatrix}=0$$

In particular, ϑ is a viscosity super-solution of:

$$-\frac{\partial^2\vartheta}{\partial z^2}\geq 0.$$

$\Lambda^+(J)$ denotes the biggest eigenvalue of J.

$$\Lambda^+(M) = 0 \Leftrightarrow \max_{||\alpha||=1} \alpha^T M\alpha = 0.$$

The HJB equation can be rewritten:

$$\max_{{\alpha_1}^2+{\alpha_2}^2=1} \left\{ -{\alpha_1}^2 \frac{\partial v}{\partial t} - {\alpha_1}^2 b(t,x)\frac{\partial v}{\partial x} - \frac{1}{2}\mathrm{tr}[a(t,x,\alpha)D^2 v] \right\} = 0.$$

$$a(t,x,\alpha) := \begin{pmatrix} \sigma^2(t,x){\alpha_1}^2 & \alpha_1\alpha_2\sigma(t,x) \\ \alpha_1\alpha_2\sigma(t,x) & {\alpha_2}^2 \end{pmatrix}.$$

Remark: α_1 could vanish.

- If $\alpha_1 = 0 \Rightarrow$ Elliptique equation.
- If $\alpha_1 \neq 0 \Rightarrow$ Parabolic equation.

$$\Lambda^+(M) = 0 \Leftrightarrow \max_{||\alpha||=1} \alpha^T M\alpha = 0.$$

The HJB equation can be rewritten:

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$$a(t,x,\alpha) := \begin{pmatrix} \sigma(t,x)\alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \sigma(t,x)\alpha_1 \\ \alpha_2 \end{pmatrix}^T$$

Remark: α_1 could vanish.

- If $\alpha_1 = 0 \Rightarrow$ Elliptique equation.
- If $\alpha_1 \neq 0 \Rightarrow$ Parabolic equation.

State-constrained stochastic problems: general setting

Come back to the general case: uniqueness result

Boundary conditions for $z \leq 0$:
Let w_0 be the value of the following unconstrained problem:

$$w_0(t,x) := \inf_{u \in \mathcal{U}} \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T d(X^u_{t,x}(s), \mathcal{K})\, ds)\right].$$

Proposition (A Dirichlet boundary condition)

For any $z \leq 0$, $\quad w(t,x,z) \equiv w_0(t,x) - z.$

Proof: For any $z \in \mathbb{R}$,

$$\begin{aligned} & w(t,x,z) \\ & \geq \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\Big[\psi(X^u_{t,x}(T)) - z - \underbrace{\int_t^T \alpha_s dB_s}_{\mathbb{E}(.)\equiv 0} + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\Big] \\ & \geq w_0(t,x) - z \end{aligned}$$

State-constrained stochastic problems: general setting

Come back to the general case: uniqueness result

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State-constrained stochastic problems: general setting

Come back to the general case: uniqueness result

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Proof: For any $z \in \mathbb{R}$,

$$w(t,x,z)$$
$$\ge \inf_{(u,\alpha)\in\mathcal{U}\times\mathcal{A}} \mathbb{E}\left[\psi(X^u_{t,x}(T)) - z - \underbrace{\int_t^T \alpha_s dB_s}_{\mathbb{E}(.)\equiv 0} + \int_t^T g_{\mathcal{K}}(X^u_{t,x}(s))ds\right]$$
$$\ge w_0(t,x) - z$$

State-constrained stochastic problems: general setting

Step 3: HJB equation and uniqueness result

Proof (continued): On the other hand, for $z \le 0$, choosing the particular control $\alpha_s :\equiv 0$:

$$
\begin{aligned}
&w(t,x,z)\\
&\qquad \le \inf_{u\in\mathcal{U}} \mathbb{E}\left[\max(\underbrace{\psi(X^u_{t,x}(T)) - z}_{\ge 0},\ 0) + \int_t^T g_{\mathcal{K}}(.)ds\right]\\
&\qquad \le w_0(t,x) - z
\end{aligned}
$$

State-constrained stochastic problems: general setting

Step 3: HJB equation and uniqueness result

Theorem (Comparison principle, Dirichlet case (Bokanowski-Picarelli-HZ'15))

The value function is the unique viscosity solution to the following HJB equation

$$\sup_{u \in U} \Lambda^{+}\Big(\mathcal{L}^{u}(x, z, \partial_t w, Dw, D^2 w)\Big) = 0, \quad t < T,\ x \in \mathbb{R}^d,\ z \geq 0,$$

$$w(T, x, z) = g_{\psi}(x, z) \equiv \max(\psi(x) - z, 0), \quad x \in \mathbb{R}^d, z \geq 0,$$

$$w(t, x, 0) = w_0(t, x), \quad t < T, x \in \mathbb{R}^d$$

w has linear growth in x ($|w| \leq C(1 + |x|)$)

Conclusion

- state-constrained OCP can be recasted into a state-constrained reachability problem, adding
  - a state variable
  - an $\mathbb{R}^p$-valued unbounded control
- the state-constrained reachability problem can be modelled by a level set approach and an auxiliary unconstrained OCP
- The value of this OCP is characterized as the unique solution of a "special" HJB equation. It is continuous and can be approximated by a large panel of numerical methods (finite differences, Semi-Lagrangian methods, Markov Chain approximations, ...).

Thanks for your attention!!