Optimality of refraction strategies for Lévy processes

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Outline

- Introduction: Motivation and framework
- Control problem
- First order conditions
- Limit results
- Singular control problem
- Numerical results

Motivation

Suppose that we have a Brownian motion W in one dimension, and we wish to control it, using controls l_t with values in the set $[0, \delta]$.

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The dynamics of the controlled process U are described as

$$U_t = W_t + \int_0^t k(l_s)ds, \quad U_0 = x,$$

for some function k.

The controller has some objective in mind

continuation...

The controller has as objective to minimize the functional

$$v(x,l.) = \mathbb{E}\left[\int_0^\infty e^{-qt}h(U_t,l_t)dt\right],$$

for a given running cost function h.

continuation...

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for a given running cost function h.

It is well known that this problem is related with the HJB equation

$$\frac{1}{2} v_{xx}(x) + \min_{l \in [0,\delta]} \{k(l) v_x(x) + h(x, l)\} - qv(x) = 0.$$

Verification...

If a smooth solution of the HJB equation can be found, we can propose as a candidate for optimal control

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$$l(x) = argmin_{l \in [0,\delta]} \{k(l) v_x(x) + h(x,l)\}.$$

- Problem: Under which conditions on the data of this optimization problem is it possible to obtain a simple solution.
- That is, on $h, \; k \; \delta, \ldots$.

Singular control

Suppose that we change now the dynamics for

$$U_t = W_t + \int_{[0,t]} dl_s, \quad U_0 = x,$$

where l_s is a nondecreasing, left continuous process with $l_0 = 0$, with an analogous structure in the cost function.

- For this problem we have some how a "simple solution", finding a solution of a free boundary problem, and reflecting the process in the boundary.
- Question: Is it possible to have something analogous for the above problem.

Controlling a Lévy processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a *spectrally negative Lévy process* $X = \{X_t; t \geq 0\}$.

The *Laplace exponent* of X is given by

$$\begin{aligned}\psi(\theta) &:= \log \mathbb{E}[\mathrm{e}^{\theta X_1}] \\ &= \gamma\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,0)} (\mathrm{e}^{\theta z} - 1 - \theta z \mathbf{1}_{\{z>-1\}})\nu(\mathrm{d}z), \quad \theta \geq 0.\end{aligned}$$

Here ν is a Lévy measure with support in $(-\infty, 0)$ and satisfying the integrability condition

$$\int_{(-\infty,0)} (1 \wedge z^2)\nu(\mathrm{d}z) < \infty.$$

Remarks on X

- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} |z|\, \nu(\mathrm{d}z) < \infty$
- In this case, we write the Laplace exponent as

$$\psi(\theta) = \tilde{\gamma}\theta + \int_{(-\infty,0)} (\mathrm{e}^{\theta z} - 1)\nu(\mathrm{d}z), \quad \theta \geq 0,$$

  with $\tilde{\gamma} := \gamma - \int_{(-1,0)} z\, \nu(\mathrm{d}z)$.
- We exclude the case in which X is the negative of a subordinator (i.e., X has monotone paths a.s.). This assumption implies that $\tilde{\gamma} > 0$ when X is of bounded variation.
- Let $\mathbb{F} := \{\mathcal{F}_t; t \geq 0\}$ be the filtration generated by X.

Formulation of the control problem

- Fix $\beta \in \mathbb{R}$, $\delta > 0$ and a measurable function $h : \mathbb{R} \to \mathbb{R}$.

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- Define Π_δ as the set of *absolutely continuous strategies* π given by adapted processes $L_t^\pi = \int_0^t \ell_s^\pi \mathrm{d}s$, $t \geq 0$, with ℓ^π restricted to take values in $[0, \delta]$ uniformly in time.

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- For $q > 0$ fixed, the objective is to consider the net present value (NPV) of the expected total costs

$$v_\pi(x) := \mathbb{E}_x \Big[\int_0^\infty \mathrm{e}^{-qt} (h(U_t^\pi) + \beta \ell_t^\pi) \mathrm{d}t \Big]. \qquad (1)$$

Objectives

The state process is

$$U_t^\pi := X_t - L_t^\pi, \quad t \geq 0,$$

and the objective is to compute the (optimal) value function

$$v(x) := \inf_{\pi \in \Pi_\delta} v_\pi(x), \quad x \in \mathbb{R}, \tag{2}$$

as well as the optimal strategy that attains it, if such a strategy exists.

Hypotheses

1. When X is of bounded variation, we assume that $\tilde{\gamma} - \delta > 0$.
2. We assume that there exists $\bar{\theta} > 0$ such that $\int_{(-\infty,-1]} \exp(\bar{\theta}|z|)\nu(\mathrm{d}z) < \infty$.
3. We assume h is convex and has at most polynomial growth in the tail. That is to say, there exist $m, k > 0$ and $N \in \mathbb{N}$ such that $h(x) \leq k|x|^N$ for all $x \in \mathbb{R}$ such that $|x| > m$.

Remarks and equivalent problem

- The drift-changed Lévy process

$$Y_t := X_t - \delta t, \quad t \geq 0, \tag{3}$$

  is the resulting controlled process if ℓ^π is uniformly set to be the maximal value δ, and is again a spectrally negative Lévy process.
- The cost function v_π as in (1) is well-defined and finite for all $\pi \in \Pi_\delta$.

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- The cost function v_π as in (1) is well-defined and finite for all $\pi \in \Pi_\delta$.
- We can also consider a version of this problem where a linear drift is added to the increments of X (as opposed to be subtracted): one wants to minimize, for some $\tilde{\beta} \in \mathbb{R}$, the NPV

$$\tilde{v}_\pi(x) := \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt}(h(X_t + L_t^\pi) + \tilde{\beta}\ell_t^\pi)\mathrm{d}t\Big].$$

Continuation...

▷ Claim: This problem is equivalent to the problem described above.

- We use Y as in (3) and set $\tilde{L}^{\pi}_t := \delta t - L^{\pi}_t$

- Then, we can write

$$\tilde{v}_{\pi}(x) = \mathbb{E}_x\Big[\int_0^{\infty} \mathrm{e}^{-qt}(h(Y_t - \tilde{L}^{\pi}_t) - \tilde{\beta}\tilde{\ell}^{\pi}_t)\mathrm{d}t\Big] + \frac{\tilde{\beta}\delta}{q}.$$

Hence it is equivalent to solving our problem for $\beta := -\tilde{\beta}$

Problems in mind

- X_t may represent the inventory level of some company.
- The objective of the company can be to maintain the inventory level around some target $\hat{x}$.
- The running cost function h can be used to penalized the distance between X_t and $\hat{x}$.
- The inventory can be on commodity products, such as oil, coal, water, etc.
- Inventory of shares in a particular company held by a specialist who is responsible for trading in that company's shares. An impact of selling in the asset's price can also be included.

Refraction strategies

Say $\pi^b \in \Pi_\delta$, under which the controlled process becomes the refracted Lévy process $U^b = \{U_t^b; t \geq 0\}$, with a suitable choice of the refraction boundary $b \in \mathbb{R}$. This is a strong Markov process given by the unique strong solution to the SDE

$$\mathrm{d}U_t^b = \mathrm{d}X_t - \delta \mathbf{1}_{\{U_t^b > b\}} \mathrm{d}t, \quad t \geq 0.$$

▷ U^b progresses like X below the boundary b while it does like Y above b.

▷ the total costs associated to π^b is

$$v_b(x) := \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt}(h(U_t^b) + \beta\delta \mathbf{1}_{\{U_t^b > b\}})\mathrm{d}t\Big], \quad x \in \mathbb{R}. \quad (4)$$

Introduction to scale functions

▷ The NPV (4) can be expressed in terms of the scale functions of the two spectrally negative Lévy processes X and Y.

▷ We use $W^{(q)}$ and $\mathbb{W}^{(q)}$ for the scale functions of X and Y, respectively.

▷ These are mappings from $\mathbb{R}$ to $[0, \infty)$ that take value zero on the negative half-line, while on the positive half-line they are strictly increasing functions that are defined by their Laplace transforms:

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x &= \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q), \\ \int_0^\infty \mathrm{e}^{-\theta x} \mathbb{W}^{(q)}(x)\mathrm{d}x &= \frac{1}{\psi(\theta) - \delta\theta - q}, \quad \theta > \varphi(q), \end{aligned} \tag{5}$$

Continuation...

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$$

and

$$\varphi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) - \delta\lambda = q\}.$$

▷ By the strict convexity of ψ, we derive the strict inequality $\varphi(q) > \Phi(q) > 0$.

Resolvent measure

$$R_b(x,B) := q^{-1}\mathbb{P}_x\{U^b_{\mathrm{e_q}} \in B\} = \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt}\mathbf{1}_{\{U^b_t\in B\}}\mathrm{d}t\Big], \quad B \in \mathcal{B}(\mathbb{R}$$

admits a density

$$R_b(x,\mathrm{d}y) = (r_b^{(1)}(x,y) + r_b^{(2)}(x,y)\mathbf{1}_{\{x>b\}})\mathrm{d}y, \quad y \in \mathbb{R}, \qquad (6)$$

given in terms of the scale functions.

We can also write

$$v_b(x) = v_b^{(1)}(x) + v_b^{(2)}(x)\mathbf{1}_{\{x>b\}}, \qquad (7)$$

First order condition

$$\frac{\partial}{\partial b} v_b(x) = u_b(x), \tag{8}$$

where

$$u_b(x) := \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt} h'(U_t^b)\mathrm{d}t\Big] - v_b'(x), \quad x \neq b.$$

Note: The first-order condition $\partial v_b(x)/\partial b|_{b=b^*} = 0$ is a necessary condition for the optimality of the refraction strategy π^{b^*}. Then, if such b^* exists,

$$v_{b^*}'(x) = \mathbb{E}_x[\int_0^\infty \mathrm{e}^{-qt} h'(U_t^{b^*})\mathrm{d}t].$$

Preliminary results

Proposition

For all $x, b \in \mathbb{R}$ *such that* $x \neq b$,

$$u_b(x) = \Big[\frac{\varphi(q) - \Phi(q)}{\delta\Phi(q)} \mathrm{e}^{\Phi(q)(x-b)} + \mathbf{1}_{\{x>b\}}(M(x;b) - \mathbb{W}^{(q)}(x-b))\Big] \cdot I(b).$$

Example: For the case $h(y) := \alpha y^2$, $y \in \mathbb{R}$, for some $\alpha > 0$,

$$b^* = \beta q/(2\alpha) + \mathbb{E}(-\underline{X}_{\mathrm{e_q}}) - \varphi(q)^{-1}.$$

Continuation...

In this case,

$$I(b) = 2\alpha \frac{\varphi(q) - \Phi(q)}{\varphi(q)} \int_0^{\infty} (y+b) \mathrm{e}^{-\varphi(q)y} \mathrm{d}y + \delta\Big[2\alpha \int_{-\infty}^{0} (y+b) \int_0^{\infty} \mathrm{e}^{-\varphi(q)z} \Theta^{(q)}(z-y) \mathrm{d}z \mathrm{d}y - \beta \frac{\Phi(q)}{\varphi(q)} \Big].$$

Here,

$$\frac{\varphi(q) - \Phi(q)}{\varphi(q)} \int_0^{\infty} (y+b) \mathrm{e}^{-\varphi(q)y} \mathrm{d}y = \frac{\varphi(q) - \Phi(q)}{\varphi(q)} \Big(\frac{1}{\varphi(q)^2} + \frac{b}{\varphi(q)} \Big).$$

Another example

For the case

$$h(y) := \alpha y, \quad y \in \mathbb{R},$$

for some $\alpha \in \mathbb{R}$, we have $b^* = -\infty$ when

$$\alpha/q > \beta$$

and $b^* = \infty$ otherwise.

Let Γ be the operator acting on sufficiently smooth functions f, defined by

$$\Gamma f(x) := \gamma f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{(-\infty,0)} [f(x+z) - f(x) - f'(x) z \mathbf{1}_{\{-1<z<0\}}] \nu(\mathrm{d}z).$$

Lemma (Verification)

Suppose a strategy $\hat{\pi} \in \Pi_\delta$ is such that $v_{\hat{\pi}}$ is sufficiently smooth on $\mathbb{R}$ and satisfies

$$\begin{cases} (\Gamma - q) v_{\hat{\pi}}(x) + h(x) \geq 0 & \text{if } v'_{\hat{\pi}}(x) \leq \beta, \\ (\Gamma - q) v_{\hat{\pi}}(x) - \delta(v'_{\hat{\pi}}(x) - \beta) + h(x) \geq 0 & \text{if } v'_{\hat{\pi}}(x) > \beta. \end{cases} \quad (9)$$

Then $\hat{\pi}$ is an optimal strategy and $v(x) = v_{\hat{\pi}}(x)$ for all $x \in \mathbb{R}$.

Applying the previous result

▷ It suffices to show that the function v_{b^*} is sufficiently smooth and satisfies (9).

▷ The function v_{b^*} is sufficiently smooth.

▷ The inequalities (9) for $v_{\hat{\pi}} = v_{b^*}$ hold if and only if

$$\begin{cases} v'_{b^*}(x) \geq \beta & \text{if } x > b^*, \\ v'_{b^*}(x) \leq \beta & \text{if } x \leq b^*. \end{cases} \tag{10}$$

▷ The function v_{b^*} is convex.
▷ The function v_{b^*} satisfies (9).

Main result

Theorem

The strategy π^{b^*} *is optimal and the value function is given by* $v(x) = v_{b^*}(x)$ *for all* $x \in \mathbb{R}$.

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Together with the analysis when $\delta \to \infty$.

Main result

Theorem

The strategy π^{b^*} is optimal and the value function is given by $v(x) = v_{b^*}(x)$ for all $x \in \mathbb{R}$.

Together with the analysis when $\delta \to \infty$. Recall that

$$\begin{aligned} \tilde{v}(x;\delta) &:= \inf_{\pi \in \Pi_\delta} \mathbb{E}_x\Big[\int_0^\infty \mathrm{e}^{-qt}(h(Y_t + L_t^\pi) + \tilde{\beta}\ell_t^\pi)\mathrm{d}t\Big] \\ &= v(x;\delta,-\tilde{\beta}) + \frac{\tilde{\beta}\delta}{q}, \end{aligned} \tag{11}$$

where $v(x;\delta,-\tilde{\beta})$ is the value function (2) obtained previously with X_t replaced with $X_t^{(\delta)} := Y_t + \delta t$ and β with $-\tilde{\beta}$.

Limit problem

▷ Let Π_∞ be the set of admissible strategies consisting of all right-continuous, nondecreasing and adapted processes L^π with $L^\pi_{0-} = 0$.

▷

$$\tilde{v}(x;\infty) := \inf_{\pi \in \Pi_\infty} \mathbb{E}_x \Big[\int_{[0,\infty)} \mathrm{e}^{-qt} (h(Y_t + L^\pi_t)\mathrm{d}t + \tilde{\beta} \mathrm{d}L^\pi_t) \Big]$$

▷ The infimum is attained by the *reflected Lévy process* $Y_t + L^{b^*(\infty)}_t$ with

$$L^{b^*(\infty)}_t := \sup_{0 \le t' \le t} ((b^*(\infty)) - Y_{t'}) \vee 0, \quad t \ge 0.$$

Continuation...

The lower boundary $b^*(\infty)$ is defined as the unique root of $I_\infty(b) = 0$ where

$$I_\infty(b) := \int_0^\infty h'(y+b)\mathrm{e}^{-\varphi(q)y}\mathrm{d}y + \tilde{\beta}\frac{q}{\varphi(q)}, \quad b \in \mathbb{R}. \qquad (12)$$

Continuation...

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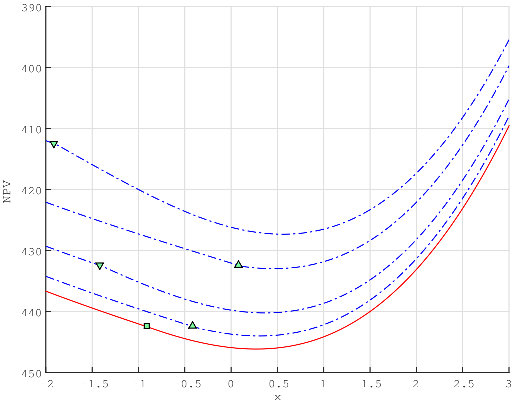
Summarizing: We show the convergences of $b^*(\delta)$ to $b^*(\infty)$

and $\tilde{v}(x;\delta)$ to $\tilde{v}(x;\infty)$ as $\delta \uparrow \infty$.

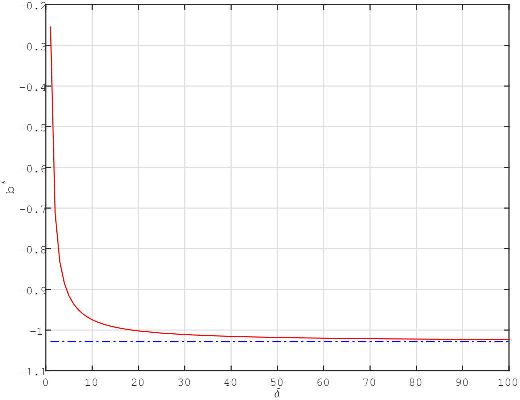
Numerical results

We focus in the case $h(x) = x^2$, with $q = .05$ and for the size type distribution we approximate a Weibull random variable. Plots of $v_b(x)$ for the cases $\beta = 5$. Each panel shows $v_{b^*}(x)$ (solid) in comparison to $v_b(x)$ (dotted) for different values of β



Continuation...

Plots of convergence as $\delta \to \infty$.



Related work

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Thank you for your attention