Optimization and Randomization Approaches to Games and Teams

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Overview

	Problems	
	Deterministic problems from **game theory**	Stochastic problems from **team theory**
Applications	Electricity markets, transportation networks etc	Decentralized control theory, information theory
Nature	Deterministic	Stochastic
Interest	Algorithms	Bounds, fundamental limits
Approach	Randomization	Optimization, de-randomization

- Interplay between convex analytic geometry and probability

Fast approximations to high-dimensional AVIs and LCPs

- Vast amounts of data is being created and stored
- **High dimensional problems** $\Longrightarrow$ exact algorithms expensive
- **Speed v/s accuracy tradeoff**
- Need for quick approximations, even if at the expense of accuracy

Standard approach

- Solve the given problem, but to lesser accuracy
- Use calculus or relaxations to construct an approximate problem

Contribution

- A new method for obtaining fast approximation

Problem class: LCPs and AVIs

Linear Complementarity Problem: $\mathrm{LCP}(M, q)$

Given $M \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n$, find x such that

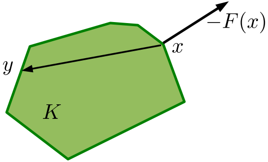
$$0 \leq x \perp Mx + q \geq 0.$$

Affine Variational Inequality: $\mathrm{AVI}(K, M, q)$

Given a **polyhedron** $K \subseteq \mathbb{R}^n$ and an affine mapping $F(x) \equiv q + Mx$, $\mathrm{AVI}(K, M, q)$, is to find a vector $x \in K$ such that

$$(y - x)^\top (Mx + q) \geq 0, \quad \forall y \in K.$$

- $\mathrm{LCP}(M, q) \equiv \mathrm{AVI}(\mathbb{R}^n_+, M, q)$
- Known to be **NP-hard**
- Includes QPs, equilibria of games etc



Problem Definition (compact K)

Approximate solution $\mathrm{AVI}(K, M, q)$

x is an $\hat{\epsilon}$-approximate solution, $\hat{\epsilon} = \hat{\epsilon}(\textit{problem constants}) > 0$, if

$$(y - x)^T (q + Mx) \geq -\hat{\epsilon}, \ \forall y \in K$$

Desiderata: We seek an **approximation algorithm** that satisfies the following requirements

- The algorithm must involve solving a lower (preferably $o(n)$) dimensional problem
- The lower dimensional problem should be **also be an AVI** (or some canonical problem)
- The **guarantees** for the algorithm **need not be deterministic**

Approach overview

- Given $\mathrm{AVI}(K, M, q)$
- **Randomly "project" (suitably)** it to create a random lower dimensional $\mathrm{AVI}(\widetilde{K}, \widetilde{M}, \widetilde{q})$
- Solve the lower dimensional problem $\longrightarrow \widetilde{x}$
- Obtain a random approximation to the solution of the given AVI

Main claim

- The **(deterministic) preimage** (under the random projection) of $\widetilde{x}$ approximately solves the given AVI.

Main tool

- Johnson-Lindenstrauss lemma

The Johnson-Lindenstrauss Lemma

Lemma (Preservation of Multiple Norms)

Let $R \in \mathbb{R}^{n \times k}$ be a uniformly random orthonormal matrix, and let $f(u) = \sqrt{\frac{n}{k}} R^\top u$ for $u \in \mathbb{R}^n$. Then for any $0 < \epsilon < 1$ and $u_1, \cdots, u_m$,

$$P(|\|u_i\|^2 - \|f(u_i)\|^2| \leq \epsilon \|u_i\|^2 \quad \forall i = 1, \cdots, m) \geq 1 - 2me^{-(\epsilon^2/2 - \epsilon^3/3)k/2}$$

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Lemma (Preservation of Multiple Inner Products)

Let $u_i, v_i, i = 1, \cdots, m$ *be a set of points in* $\mathbb{R}^n$*. For* $0 < \epsilon < 1$*,*

$$P\left(|u_i^\top v_i - f(u_i)^\top f(v_i)| \le \epsilon \|u_i\| \|v_i\| \ \forall i = 1, \cdots, m\right) \ge 1 - 4me^{-(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})k/2}$$

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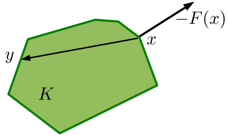
- $1 - 4me^{-(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})k/2} \ge 1 - \delta \iff k \approx \frac{1}{\epsilon^2} \ln m/\delta$.
- Only **finitely many** inner products allowed
- The dimension n appears nowhere; instead m appears
- ϵ can be chosen independently of n, m
- Notice the scaling/normalization

Key idea

- Solving an AVI amounts to finding an x s.t. $Mx + q$ makes certain angles/inner products with $y - x$ as y ranges over K



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$$P(|\underbrace{u_i^\top v_i}_{=\mathbb{E}[f(u_i)^\top f(v_i)]} - f(u_i)^\top f(v_i)| \leq \epsilon \|u_i\| \|v_i\| \;\; \forall i = 1, \cdots, m) \geq 1 - 4me^{-(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})k/2}$$

- **General idea:** Craft a lower dimensional problem such that:

  $u_i^\top v_i \approx$ optimality for higher dimensional problem

  $f(u_i)^\top f(v_i) \approx$ optimality for lower dimensional problem

- **But there is a catch!** – does not lead to an AVI as a lower-dimensional problem
- Need a **correction, adds an error**; need careful crafting to minimize error

Challenges

Convex analysis

- Simultaneously minimize the lower dimension and error
- AVIs involve **infinitely many** inner products; need to **reduce to finitely many**
- Make sure they are **few enough** – exploit the convex geometry

Solution recovery

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Solution recovery

- How to **back project** from the lower dimension?
- **Back projection** error? How to drive this to zero?

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- Will there be **tradeoffs** or can we achieve everything?

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Solution recovery

- How to **back project** from the lower dimension?
- **Back projection** error? How to drive this to zero?
- Will there be **tradeoffs** or can we achieve everything?
- **What we would like:** for fixed problem constants, we can compute tolerance. For fixed tolerance, lower dimension must be $o(n)$. As $n \to \infty$ error is within tolerance. In short no tradeoffs.

Algorithm (Compact K) I

1. Fix the error parameter $\epsilon \in (0,1)$ and the success confidence parameter $\delta \in [0,1)$. Pick the lower dimension value k such that $k \geq \frac{2\ln(\frac{4\eta}{\delta})}{(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})}$, where $\eta = |ext(K)|$.
2. Construct $n \times k$ random orthonormal matrix R
3. Construct lower dimensional $\mathrm{AVI}(\widetilde{K}, \widetilde{M}, \widetilde{q})$. Define

$$\widetilde{q} = \sqrt{\frac{n}{k}} R^\top q \quad \widetilde{M} = cR^\top MR \quad \widetilde{K} = \left\{ \widetilde{x} \in \mathbb{R}^k | \widetilde{x} = \sqrt{\frac{n}{k}} R^\top x, Ax \leq b \right\}$$

4. Solve $\mathrm{AVI}(\widetilde{K}, \widetilde{M}, \widetilde{q}) \longrightarrow \widetilde{x}$.
5. Solve the L_1 minimization problem

$$\min_{x \in \mathbb{R}^n} \quad \|x\|_1 \qquad \longrightarrow \qquad x^*$$
$$\text{s.t.} \quad \sqrt{\frac{n}{k}} R^\top x = \widetilde{x}$$

6. Project x^* on K,

$$x^\# = \underset{x \in K}{\arg\min} \quad \|x^* - x\|^2$$

7. $x^\#$ is the final output vector.

Analysis

$\widetilde{x} \longleftarrow \mathrm{AVI}(\widetilde{K}, \widetilde{M}, \widetilde{q})$
Let x_o be a deterministic pre-image such that $\sqrt{\frac{n}{k}} R^\top x_o = \widetilde{x}$.

$$(\widetilde{y} - \widetilde{x})^\top (\widetilde{q} + \widetilde{M}\widetilde{x}) \geq 0, \quad \forall \widetilde{y} \in \widetilde{K} \quad (\textit{infinitely many})$$

$$\implies \sqrt{\frac{n}{k}} (R^\top (y - x_o))^\top \sqrt{\frac{n}{k}} R^\top (q + M' x_o) \geq 0, \quad \forall y \in K \quad (M' = cMRR^\top)$$

$$\implies \sqrt{\frac{n}{k}} (R^\top (x_e - x_o))^\top \sqrt{\frac{n}{k}} R^\top (q + M' x_o) \geq 0, \quad \forall x_e \in ext(K) \quad (\textit{finite!})$$

$M' = cMRR^\top$. Preserving $\eta = |ext(K)|$ inner products, **JL lemma** $\implies$ w.p $\geq (1 - 4\eta e^{-(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})k/2})$, we have

$$(x_e - x_o)^\top (q + Mx_o) \geq \frac{n}{k} (x_e - x_o)^\top RR^\top (M - M') x_o + (-\epsilon \|x_e - x_o\| \|q + Mx_o\|) \quad \forall x_e \in ext(K).$$

Error Bounds

Since the error depends on $(M - M')$ it is tempting to take $M' = MRR^\top$ (i.e., $c = 1$).

some lines of bounding arguments later

$$\implies (y - x_o)^\top (q + Mx_o) \geq -\frac{\left(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3}\right)n}{\ln\left(\frac{4\eta}{\delta}\right)} \|M\| \|x_o\| - \epsilon \cdot D\|q\| - \epsilon \cdot D\|M\| \|x_o\|, \ \forall y \in K$$

Remember:

- There are several random x's that satisfy $\sqrt{\frac{n}{k}} R^\top x = \widetilde{x}$.
- The above result pertains to only (and all) $x = x_o$'s that are deterministic.

Finding x_o

- The algorithm gives us $\widetilde{x}$ (solution of lower dim problem)
- Need to backproject to find a **deterministic** $x_o \in \mathbb{R}^n$ from $\widetilde{x} \in \mathbb{R}^k$
- $\implies$ Problem from *compressed sensing*: recovering a "signal" from its "partial observations"
- Many results which provide "high probability" guarantees on **exact** recovery of the sparsest x_o. The result below is requires no sparsity.

Theorem ([CT06])

Suppose that $f \in \mathbb{R}^n$ obeys $\|f\|_1 \leq C_1$, and we are given k uniformly random orthonormal measurements $y = \sqrt{\frac{n}{k}} R^\top f$. Then with probability 1, we have a unique minimizer $f^\#$ to the following problem:

$$\min_{x \in \mathbb{R}^n} \quad \|x\|_1 \qquad \text{such that } y = \sqrt{\frac{n}{k}} R^\top x \quad .$$

Furthermore, with probability at least $1 - O(n^{-1/\alpha})$, we have the approximation

$$\|f - f^\#\| \leq C \cdot C_1 \cdot (\ln(n)/k)^{1/2}.$$

Here, C is a fixed constant depending on α but not on anything else. The implicit constant in $O(n^{-1/\alpha})$ is allowed to depend on α.

First result ($c = 1$) [PK14a] [PK14b]

Theorem

1. $\widetilde{x}$ *exists w.p. 1*
2. *For every* $\epsilon \in (0,1)$, *if the lower dimension* k *satisfies* $k \geq \frac{2\ln(\frac{4\eta}{\delta})}{(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})}$, $0 \leq \delta < 1$, *then with a probability of at least* $(1-\delta)$, *the deterministic pre-image* x_o *satisfies,*

$$(y - x_o)^\top (q + Mx_o) \geq \hat{\epsilon}, \ \forall y \in K$$

where $\hat{\epsilon} = -\frac{(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})n}{2ln(\frac{4\eta}{\delta})} \|M\| \|x_o\| - \epsilon \cdot D\|q\|_2 - \epsilon \cdot D\|M\| \|x_o\|.$

3. *With a probability at least* $1 - O(n^{-1/\alpha})$:

$$\|x_o - x^{\#}\| \leq C \cdot \|x_o\|_1 \cdot (ln(n)/k)^{1/2}$$

where $\alpha > 0$ *is a sufficiently small number and* C *is a constant depending only on* α.

Revisiting the bound

- Lower dimension = $O(\ln \eta)$ where $\eta = |ext(K)|$
- **Good:** reveals something combinatorial about the problem
- **Bad:** requires K to be a polytope, does not cover LCPs
- **Bad:** worst case $\ln \eta \approx n$.
- **Unclear:** is this the true complexity?
- **Error:** Can the error be made smaller?

Revisiting the bound

- Taking $c = 1 \iff \widetilde{M} = R^\top MR \iff$ pseudoinverse – **Too conservative.**
- Turns out: $c = \frac{n}{k}$ can give **lower error**, but with needs preservation of **more inner products**
- But not too many more!
- **Key idea:** Matrix-vector inner product preservation

$$P\left(\| \frac{n}{k} d^\top RR^\top A - d^\top A \| \leq \epsilon \|d\| \|A\|_F \right) \geq 1 - 4ne^{-(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})k/2},$$

Improved result

Theorem

Let $c = \frac{n}{k}$. For every $\epsilon \in (0,1)$ and $0 < \delta \leq 1$, if the lower dimension k satisfies $k \geq 2\ln(\frac{4(n+n\eta+\eta)}{\delta})/(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})$, then with a probability of at least $(1-\delta)$, the deterministic vector x_o solves $\mathrm{AVI}(q,\ M,\ K)$ approximately, i.e.,

$$(y - x_o)^\top (q + Mx_o) \geq -\hat{\epsilon},\ \ \forall y \in K$$

where $\hat{\epsilon} = \epsilon\sqrt{1 + 2\epsilon + \epsilon^2 n}\|M\|_F\|x_o\| + \epsilon \cdot D\|q\|_2 + \epsilon \cdot D\|M\|\|x_o\|$.

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where $\hat{\epsilon} = \epsilon\sqrt{1 + 2\epsilon + \epsilon^2 n}\|M\|_F\|x_o\| + \epsilon \cdot D\|q\|_2 + \epsilon \cdot D\|M\|\|x_o\|$.

Corollary

With $\epsilon = \frac{\epsilon'}{n^{\frac{1}{4}}}$, we get $\hat{\epsilon} = -\epsilon' \cdot const(M, q, K)$ and $k \sim \frac{1}{(\epsilon')^2}\sqrt{n}\ln(\frac{4(n+n\eta+\eta)}{\delta})$.

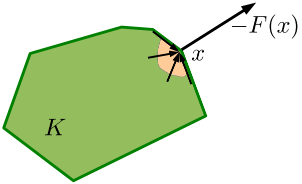
- ϵ' can be selected based on only the constants in the problem
- $k = o(n)$ and $\frac{\ln n}{k} \to 0$. Thus back projection error also $\to 0$

Further improvement [PK15]

- How large is $\eta = |ext(K)|$? If $K = \{x | Ax = b, x \geq 0\}$ and $A \in \mathbb{R}^{m \times n}$ is full row rank, $\eta \leq \binom{n}{m}$ In other representations, there is no easy bound.
- Analysis required compactness of K. What if K is unbounded?

Tangent cone

$$\mathcal{T}(x;K) = \left\{ d \mid \exists \{x_k\} \subseteq K, x_k \to x, \tau_k \downarrow 0, \text{ such that } d = \lim_{k\to\infty} \frac{x_k - x}{\tau_k} \right\}.$$



$$x \text{ solves AVI}(K, M, q) \iff (Mx+q)^\top d \geq 0 \qquad \forall d \in \underbrace{\mathcal{T}(x;K)}_{\textit{again infinite!}}.$$

Further improvement

Suffices to preserve inner products with only the generators of the tangent cone

Lemma

If K is polyhedral, $\mathcal{T}(x;K)$ is polyhedral. A point $x \in K$ solves $\mathrm{AVI}(K, M, q)$ if and only if

$$d^\top F(x) \geq 0, \qquad \forall d \in conv\left(ext(\mathcal{T}(x;K))\right).$$

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Lemma

Let K be a polyhedron, $x \in K, \widetilde{x} = \sqrt{\frac{n}{k}} R^\top x$ and let $\widetilde{K} = \sqrt{\frac{n}{k}} R^\top K$. Then,

$$\sqrt{\frac{n}{k}} R^\top \mathcal{T}(x;K) = \mathcal{T}(\widetilde{x};\widetilde{K}).$$

- Proceed as before ...

$$(\tilde{M}\tilde{x} + \tilde{q})^\top \widetilde{d} \geq 0 \qquad \forall \widetilde{d} \in \underbrace{\mathcal{T}(\widetilde{x};\widetilde{K})}_{\sqrt{\frac{n}{k}} R^\top \mathcal{T}(x_o;K) \implies \sqrt{\frac{n}{k}} R^\top ext(\mathcal{T}(x_o;K))}$$

Applying to LCPs

High dim LCPs → low dimensional AVI

Theorem

Let $\epsilon \in (0,1), \delta \in (0,1]$ *and* $\mathrm{LCP}(M,q)$ *be provided as input to Algorithm. If the lower dimension* k *satisfies* $k \geq 2\ln\left(\frac{4}{\delta}(n^2+2n)\right)/\left(\frac{\epsilon^2}{2}-\frac{\epsilon^3}{3}\right)$, *then* **conditioned on the event that** $\widetilde{x}$ **exists** *with probability at least* $(1-\delta)$, x_o *satisfies*

$$|(q+Mx_o)_i| \leq \epsilon\|q+Mx_o\| + \epsilon\sqrt{1+\epsilon^2 n+2\epsilon}\|M\|_F\|x_o\| \quad \forall i \notin \mathcal{A}(x_o)$$

$$(q+Mx_o)_i \geq -\epsilon\|q+Mx_o\| - \epsilon\sqrt{1+\epsilon^2 n+2\epsilon}\|M\|_F\|x_o\| \quad \forall i \in \mathcal{A}(x_o)$$

$$|x_o^\top(q+Mx_o)| \leq \epsilon\|x_o\|_1\|q+Mx_o\| + \epsilon\sqrt{1+\epsilon^2 n+2\epsilon}\|x_o\|_1\|M\|_F\|x_o\|$$

where $\mathcal{A}(x) = \{i | x_i = 0\}$

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where $\mathcal{A}(x) = \{i | x_i = 0\}$

- **Conditional probability:** Lower dim AVI obtained from LCPs need not be solvable

Corollary

Can take $k \sim \frac{1}{(\epsilon')^2}\sqrt{n}\ln(n^2+2n) = o(n)$. *Back projection error* $\to 0$

AVIs on noncompact polyhedra

Theorem

Let $\epsilon, \delta \in (0,1)$ *and suppose* $\mathrm{AVI}(K, M, q)$ *is provided as input to the algorithm and let* $k \geq 2\ln\left(\frac{4}{\delta}(n + n\kappa + \kappa)\right)/(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3})$, *where*

$$\kappa = \max_{x \in K} |ext(\mathcal{T}(x; K))| = O(n)^*.$$

1. *If* $\mathrm{AVI}(K, M, q)$ *has a nondegenerate solution,* $\widetilde{x}$ *exists with high probability*
2. *Conditioned on the event that* $\widetilde{x}$ *exists, with probability greater than* $1 - \delta$, x_o *satisfies,*

$$d_e^\top (q + Mx_o) \geq -\epsilon \cdot \sqrt{1 + 2\epsilon + \epsilon^2 n} \|M\|_F \|x_o\| - \epsilon \cdot \|q + Mx_o\| \qquad \forall\ d_e \in ext(\mathcal{T}(x_o; K)).$$

- $^*ext(\bullet)$ = set of generators of $\bullet$. For a polytope K, $|ext(\mathcal{T}(x; K))| \geq |ext(K)|$. Under a suitable representation for K, $\kappa = O(n)$

Corollary

Once again, we may take $k \sim \frac{1}{(\epsilon')^2}\sqrt{n}\ln(n + n\kappa + \kappa)$.

Further remarks

Solving the lower dimensional problem

- Lower dim problem has implicit constraints
- Make implicit variables explicit – sparse problem higher dimensional problem
- Need a solver that can exploit constraints like these

Numerical performance

- Much faster; accuracy improves as k increases
- Useful for quickly deriving an initial point
- Substantial savings above random starting points

Approximate solutions for LCPs

n	k	Major (low)	Minor (low)	Major (direct)	Minor (direct)	Natural Map Residual	Inner Prod	Max Inner Prod	Min Inner Prod	ϵ'
400	50	4	36	1	342	0.95	0.57	1.05	0.00	0.005
400	60	7	233	1	342	0.95	10.04	8.91	0.00	0.005
400	70	4	19	1	342	0.96	1.22	1.44	0.00	0.006
500	70	4	25	1	196	0.93	0.98	1.45	0.00	0.004
500	80	4	4	1	196	0.98	1.21	1.25	0.00	0.005
500	90	5	13	1	196	0.96	1.30	1.31	0.00	0.005
600	70	2	3	1	111	0.98	0.39	0.51	0.00	0.004
600	80	3	6	1	111	0.94	0.79	1.07	0.00	0.004
600	90	7	11	1	111	0.97	1.51	1.93	0.00	0.004
700	75	4	22	1	655	0.94	0.33	0.45	0.00	0.003
700	100	5	28	1	655	0.96	0.70	0.63	0.00	0.004
700	125	14	86	1	655	0.96	2.14	1.35	0.00	0.004
700	150	14	348	1	655	0.98	18.58	10.49	0.00	0.004
850	100	4	419	1	166	0.95	5.41	6.01	0.00	0.003
850	125	9	1010	1	166	0.97	1.86	1.40	0.00	0.003
850	150	11	121	1	166	0.98	2.71	1.82	0.00	0.004
1000	100	6	245	1	1269	0.97	2.79	3.59	0.00	0.002
1000	125	6	141	1	1269	0.96	3.17	2.32	0.00	0.003
1000	150	13	1109	1	1269	0.97	0.90	0.72	0.00	0.003

Exact solutions of AVIs

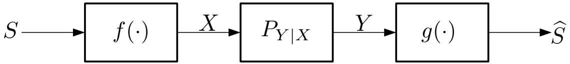
n	k	m	Major (New)	Major (Old)	Major Total	Minor (New)	Minor (Old)	Minor Total	Time (new)	Time (direct)
300	10	10	0	3	3	0	3156	13	0.07	0.33
300	50	10	1	3	5	312	3156	328	0.11	0.33
300	90	10	1	3	6	622	3156	639	0.12	0.33
300	130	10	1	3	7	312	3156	329	0.11	0.33
400	25	10	1	2	5	4	4126	18	0.19	0.55
400	50	10	1	2	5	418	4126	432	0.21	0.55
400	100	10	1	2	7	834	4126	851	0.25	0.55
400	150	10	1	2	8	834	4126	852	0.26	0.55
400	200	10	1	2	8	1246	4126	1265	0.28	0.55
600	25	10	5	46	9	8225	58025	8239	2.44	19.27
600	50	10	7	46	11	10178	58025	10193	3.03	19.27
600	100	10	6	46	12	8888	58025	8904	2.68	19.27
600	200	10	6	46	13	10155	58025	10172	2.98	19.27
800	25	10	0	2	4	1	8137	15	0.65	2.52
800	50	10	0	2	5	820	8137	835	0.81	2.52
800	100	10	1	2	6	2	8137	18	0.67	2.52
800	300	10	1	2	8	1634	8137	1652	0.99	2.52
1000	50	10	0	3	5	2	10130	17	1.14	5.86
1000	100	10	1	3	6	1011	10130	1027	1.51	5.86
1000	300	10	1	3	9	1006	10130	1039	1.44	5.86

Team decision problems

Team decision problems

- Random environment (prerealized but unknown) $\xi \in \mathbb{R}^n$
- N players
- Players get information, denoted $z_1, \cdots, z_N$; $z_i \in \mathbb{R}^{m_i}$
- They have to choose "strategies", i.e., measurable functions $\gamma_i : \mathbb{R}^{m_i} \to \mathbb{R}$ to jointly minimize
$$J(\gamma_1, \cdots, \gamma_N) = \mathbb{E}[L(u_1, \cdots, u_N; \xi)],$$
where $u_i = \gamma_i(z_i)$ and $L : \mathbb{R}^{N+n} \to \mathbb{R}$
- Complications arise due to z_i
- **Simplest case** is when $z_i = \eta_i(\xi)$. Studied first by [Rad62].
- **Hard case** is when $z_i = \eta_i(u_{S_i}, \xi)$; even simple instances are unsolved [Wit68].
Solvable subcase "classical" stochastic control. There is a time axis and
$$S_i = u_1, \cdots, u_{i-1} \quad \text{and } z_i \supseteq z_{i-1}$$
- Without this, extremely hard. Throw up fundamental issues such causality, "who knows what" etc

Introduction

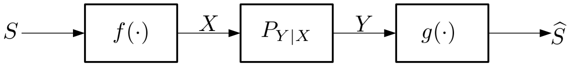


Stochastic optimal control problem with *non-classical information pattern*:

$$\text{SC} \quad \underset{f,g}{\text{minimize}} \quad \mathbb{E}[\kappa(S,X,Y,\widehat{S})]$$
$$X = f(S)$$
$$\widehat{S} = g(Y).$$

- $S, X, Y, \widehat{S}$ take values in $\mathcal{S}, \mathcal{X}, \mathcal{Y}, \widehat{\mathcal{S}}$.
- **Given:** $S \sim P_S,\ Y|X \sim P_{Y|X}$.
- **Information structure:** $S \to X \to Y \to \widehat{S}$
- **Cost:** $\kappa : \mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \widehat{\mathcal{S}} \to \mathbb{R}$
- **Decision variables:** $f : \mathcal{S} \to \mathcal{X},\ g : \mathcal{Y} \to \widehat{\mathcal{S}}$

Examples and challenges



- **Witsenhausen:**

$$Y = X + W, \quad S, W \sim \text{Gaussian}, \quad \kappa(S, X, Y, \widehat{S}) \equiv (S - X)^2 + k(\widehat{S} - X)^2$$

- **Finite block-length channel coding:**

$$S \sim \text{Unif}(\mathcal{S}), \quad P_{Y|X} = \prod_{i=1}^{n} P_{Y_i|X_i}, \quad \kappa(S, X, Y, \widehat{S}) \equiv \mathbb{I}_{\{S \neq \widehat{S}\}}$$

- **Finite block-length lossy source-channel coding:**

$$\mathcal{S} = \prod_{i=1}^{n} \mathcal{S}_i \quad P_{Y|X} = \prod_{i=1}^{n} P_{Y_i|X_i}, \quad \kappa(S, X, Y, \widehat{S}) \equiv \frac{1}{n} \sum_{i=1}^{n} d(S_i, \widehat{S}_i)$$

- **Common fundamental challenge: Obtaining lower bounds**

Standard approach

- A block of n symbols is encoded as $(X_1, \cdots, X_n) = f(S_1, \cdots, S_n)$; this entails a natural delay
- Embed the given finite blocklength problem in an "infinite" blocklength version (equivalent to allowing arbitrary delay)
- Use results from information theory to solve the infinite blocklength problem, which bounds the given problem
- This leads to bounds that **necessarily involve** mutual information.

Lower bounds via convex relaxation

[KC12, KC15]

Original problem

$$\underset{f,g}{\text{minimize}} \quad \mathbb{E}[\kappa(S,X,Y,\widehat{S})]$$
$$X = f(S)$$
$$\widehat{S} = g(Y).$$

$\Longleftrightarrow$

Optimization over distributions

(Nonconvex!)

$$\underset{Q}{\text{minimize}} \quad \mathbb{E}[\kappa(S,X,Y,\widehat{S})]$$
$$Q_S = P_S, Q_{Y|X} = P_{Y|X}$$
$$S \to X \to Y \to \widehat{S}.$$

$\Downarrow$

A particular relaxation

$$\underset{Q}{\text{minimize}} \quad \mathbb{E}[\kappa(S,X,Y,\widehat{S})]$$
$$Q_S = P_S, \; Q_{Y|X} = P_{Y|X}$$
$$(DPI) \; I(S;\widehat{S}) \le I(X;Y)$$

$\Longleftarrow$

Convex relaxation

$$\underset{Q}{\text{minimize}} \quad \mathbb{E}[\kappa(S,X,Y,\widehat{S})]$$
$$Q \in \text{larger convex set}$$

Consequences:

- Infinite blocklength bounds are the **same** as those from convex relaxation by mutual information. This relaxation is **tight** for the variant of Witsenhausen in [BB87]
- New proofs, generalizations etc.

What is a good relaxation?

[JK15, JK16]

- **Ideal relaxation** – convex hull!
- Solutions of the original problem and of the relaxation lie on extreme points of respective feasible regions
- A good relaxation must *include all extreme points* of the original problem *as extreme points* of its feasible region
- Simplicity – solving/bounding the relaxed problem should not itself be hard

Shortcomings of the DPI relaxation

- DPI relaxation **does not have this property**
- $\implies$ There are costs for which the DPI relaxation is not tight

This talk

- A linear programming relaxation with the required property
- Show that it is tight in certain cases, derive new bounds

Linear programming relaxation

- $\mathcal{S}, \mathcal{X}, \mathcal{Y}, \widehat{\mathcal{S}}$ are finite
- Non-convexity is due to bilinear terms "$Q_{X|S}(x|s)Q_{\widehat{S}|Y}(\widehat{s}|y)$"

Relaxation:

1. Replace bilinear terms with new variables W:

$$W(s,x,y,\widehat{s}) = Q_{X|S}(x|s)Q_{\widehat{S}|Y}(\widehat{s}|y)$$

2. Add linear constraints that $W(s,x,y,\widehat{s})$ must satisfy, e.g.,

$$\sum_x W(s,x,y,\widehat{s}) = Q_{\widehat{S}|Y}(\widehat{s}|y), \qquad \forall \widehat{s}, y$$

$$\text{LP} \min_{Q_{X|S}, Q_{\widehat{S}|Y}, W} \quad \sum_z \kappa(z) P_S(s) P_{Y|X}(y|x) W(s,x,y,\widehat{s})$$

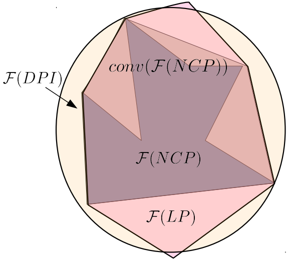
$$\text{subject to} \quad \begin{aligned}
\textstyle\sum_x Q_{X|S}(x|s) &= 1 && \forall s \\
\textstyle\sum_{\widehat{s}} Q_{\widehat{S}|Y}(\widehat{s}|y) &= 1 && \forall y \\
\textstyle\sum_x W(s,x,y,\widehat{s}) - Q_{\widehat{S}|Y}(\widehat{s}|y) &= 0 && \forall s, \widehat{s}, y \\
\textstyle\sum_{\widehat{s}} W(s,x,y,\widehat{s}) - Q_{X|S}(x|s) &= 0 && \forall x, s, y \\
Q_{X|S}(x|s) + Q_{\widehat{S}|Y}(\widehat{s}|y) - W(s,x,y,\widehat{s}) &\le 1 && \forall s, x, y, \widehat{s} \\
Q_{X|S}(x|s) &\ge 0 && \forall s, x \\
Q_{\widehat{S}|Y}(\widehat{s}|y) &\ge 0 && \forall \widehat{s}, y \\
W(s,x,y,\widehat{s}) &\ge 0 && \forall s, x, y, \widehat{s}
\end{aligned}$$

Linear programming relaxation

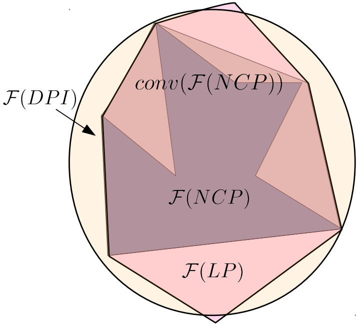
Lemma

All extreme points *of the original problem are extreme points of the LP relaxation.*

- New variables W lift the problem to a higher dimensional space. There it is easier to identify implied inequalities
- Related to techniques in combinatorial optimization – "lift and project", "reformulation-linearization", "method of moments", "McCormick Relaxations", etc.
- Our relaxation *equals the convex hull* in some extreme cases, (e.g., $|\mathcal{S}| = 1$). Exact characterization of the convex hull remains open
- Extendable to networked settings; similar ideas can be applied.



Power of the LP relaxation



1. One can **systematically obtain** lower bounds

Nonclassical problem	$\geq$	**value of LP relaxation**	$=$	**value of dual LP**	$\geq$	**value of any dual feasible point**

2. Can be **intersected** with other relaxation to improve both relaxations
3. Gives new and wider class of **inverse optimal cost functions**

Lossy source-channel coding

- Binary uniform source: $S = (S_1, \cdots S_n)$, where $S_i = \mathrm{Ber}(\frac{1}{2})$, i.i.d.
- Binary symmetric channel: $X = (X_1, \cdots, X_n)$, $Y = (Y_1, \cdots, Y_n)$, where $X_i, Y_i \in \{0, 1\}$.

$$P_{Y_i|X_i}(0|1) = P_{Y_i|X_i}(0|1) = \epsilon < \frac{1}{2}, \qquad P_{Y|X}(y|x) = \Pi_{i=1}^{n} P_{Y_i|X_i}(y_i|x_i)$$

- Cost :

$$\kappa(s, x, y, \widehat{s}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{s_i \neq \widehat{s}_i} \qquad \Longrightarrow \mathbb{E}[\kappa] = \frac{1}{n}\sum_{i=1}^{n} P(S_i \neq \widehat{S}_i)$$

Dual
$$\begin{aligned}
\underset{\gamma^A,\gamma^B,\lambda^A,\lambda^B,\mu}{\text{maximize}} \quad & \sum_s \gamma^A(s) + \sum_y \gamma^B(y) - \sum_z \mu(z) \\
\text{subject to} \quad & \gamma^A(s) - \textstyle\sum_y \lambda^B(x, s, y) - \sum_{\widehat{s}, y} \mu(z) \leq 0 & \forall x, \\
& \gamma^B(y) - \textstyle\sum_s \lambda^A(s, \widehat{s}, y) - \sum_{x, s} \mu(z) \leq 0 & \forall \widehat{s}, \\
& \lambda^A(s, \widehat{s}, y) + \lambda^B(x, s, y) + \mu(z) \leq \tfrac{1}{2^n}\epsilon^{|x-y|}(1-\epsilon)^{n-|x-y|}\tfrac{|s-\widehat{s}|}{n} & \forall z \\
& \mu(z) \geq 0 & \forall z
\end{aligned}$$

$$z := (s, x, y, \widehat{s})$$

$$|s| := \text{Hamming weight of } s$$

$$|x - y| := \text{Hamming distance between } x, y$$

Lossy source-channel coding

Theorem (Tightness for uniform source)

Let $0 < \epsilon < \frac{1}{2}$. The LP relaxation is **tight** *with cost $= \epsilon$ for all n. Moreover, there exists a dual* **optimal** *solution that takes the following form:*

$$\gamma^A(s) \equiv const, \quad \gamma^B(y) \equiv const, \quad \mu(x, s, \widehat{s}, y) \equiv 0$$
$$\lambda^A(s, \widehat{s}, y) \equiv \lambda^A(|s - \widehat{s}|), \quad \lambda^B(x, s, y) \equiv \lambda^B(|x - y|)$$

Proof.

It is easy to show that cost ϵ is achievable in the original problem. Moreover, the following point is feasible:

$$\lambda^A(0) = c \qquad \gamma^A = \sum_{k=0}^{n} {}^nC_k \lambda^B(k), \quad \gamma^B = \sum_{k=0}^{n} {}^nC_k \lambda^A(k)$$

$$\lambda^A(|s - \widehat{s}| = a) = c + \sum_{k=1}^{a} \epsilon^k (1-\epsilon)^{(n-k)} \frac{1}{n \mid S \mid} \qquad \forall a = 1, 2 \ldots n$$

$$\lambda^B(|x - y| = b) = \frac{b}{n \mid S \mid} \epsilon^b (1-\epsilon)^{(n-b)} - \lambda^A(|s - \widehat{s}| = b) \quad \forall b = 0, 1, 2, \cdots n$$

where c is any scalar. The dual cost is, $\mid S \mid \gamma^A + \mid Y \mid \gamma^B = \epsilon$. ∎

Lossy source-channel coding (binary non-uniform source)

More generally ...

- Source (possibly **non-uniform**): $S = (S_1, \cdots S_n)$, $S_i = \mathrm{Ber}(p)$ i.i.d., $p \in [0, 1]$.

Theorem (A general lower bound)

There exists a dual feasible point of the form:

$$\gamma^A(s) \equiv \gamma^A(\mathbf{s}), \quad \gamma^B(y) \equiv const, \quad \mu(x, s, \widehat{s}, y) \equiv 0,$$

$$\lambda^A(s, \widehat{s}, y) \equiv g_{\mathbf{s}}(|s - \widehat{s}|), \quad \lambda^B(x, s, y) = -g_{\mathbf{s}}(|x - y|) + |x - y| g_{\mathbf{s}}'(|x - y|).$$

where $\mathbf{s} \equiv |s|$ *and* $g_{\mathbf{s}} : \mathbb{R} \to \mathbb{R}$ *is* **any concave differentiable function** *satisfying*

$$g_{\mathbf{s}}'(x) \leq \frac{p^{\mathbf{s}}(1 - p)^{n - \mathbf{s}}}{n} \epsilon^x (1 - \epsilon)^{n - x}.$$

We have the following lower bound,

$$\frac{1}{n} \sum_{i=1}^{n} P(S_i \neq \widehat{S}_i) \geq$$

$$\sup_g \left[\sum_{\mathbf{s}=0}^{n} {}^nC_{\mathbf{s}} \sum_{k=0}^{n} {}^nC_k \left[-g_{\mathbf{s}}(k) + k g_{\mathbf{s}}'(k) \right] + 2^n \left[\min_{v \in \{0,1,\cdots n\}} \sum_{i=0}^{n-v} \sum_{j=0}^{v} g_{i-j+v}(i+j)^{n-v} C_i {}^v C_j \right] \right]$$

Channel coding

- Source: $\mathcal{S} = \widehat{\mathcal{S}} \subset \mathbb{N}$, $|\mathcal{S}| < \infty$, $S \sim \mathrm{Unif}(\mathcal{S})$
- Binary symmetric channel: $X = (X_1, \cdots, X_n)$, $Y = (Y_1, \cdots, Y_n)$, where $X_i, Y_i \in \{0, 1\}$.

$$P_{Y_i|X_i}(0|1) = P_{Y_i|X_i}(0|1) = \epsilon < \frac{1}{2}, \qquad P_{Y|X}(y|x) = \Pi_{i=1}^{n} P_{Y_i|X_i}(y_i|x_i)$$

- Cost:

$$\kappa(s, x, y, \widehat{s}) = \mathbb{I}_{s \neq \widehat{s}} \qquad \Longrightarrow \qquad \mathbb{E}[\kappa] = \mathbb{P}(S \neq \widehat{S}) = P(\text{“error''})$$

Theorem (“Strong converse”)

There exists a dual feasible point of the form,

$$\mu \equiv 0, \quad \lambda^A(s, \widehat{s}, y) \equiv \lambda^A(|s - \widehat{s}|), \quad \lambda^B(x, s, y) \equiv \lambda^B(|x - y|)$$
$$\gamma^A(s) \equiv const, \quad \gamma^B(y) \equiv const.$$

Let $|\mathcal{S}| = 2^{nR}$*. If*

$$R > 1 + \log_2(1 - \epsilon),$$

$P(S \neq \widehat{S}) \xrightarrow{n} 1$ **exponentially**, *i.e., there exists* $c \in (0, 1)$ *such that* $P(S \neq \widehat{S}) \geq 1 - c^n$.

Thank You!
Questions, comments?

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