A generalized monotone control problem with average cost criterion

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Outline

The dynamical system

Consider the system

$$\begin{cases} \dot{y}(t) = f(y(t), \alpha(t)), & t > 0, \\ y(0) = x \in \overline{\Omega} \subseteq \mathbb{R}^n, \end{cases} \tag{1}$$

where Ω is a bounded domain, regular enough, and $\alpha \in \mathcal{A}(a)$, defined as the set of controls $\alpha : \mathbb{R}_+ \to [0,1]$ satisfying

$$\begin{cases} d\alpha = g(\alpha)dt - du, \\ \alpha(0) = a \in [0,1], \end{cases} \tag{2}$$

for some non decreasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$. Here, f and g are bounded Lipschitz continuous functions and we make some additional assumptions to guarantee that y remains in $\overline{\Omega}$ and α remains in $[0,1]$.

The usual monotone control problem corresponds to $g \equiv 0$.

Optimization criteria

Given $\alpha \in \mathcal{A}(a)$, let $y_x(t,\alpha)$ be the response of (1). For a bounded Lipschitz continuous function ℓ, we consider the problems

Infinite horizon problem with discount factor $\lambda > 0$

$$J^\lambda(x,\alpha) = \int_0^\infty \ell(y_x(t,\alpha),\alpha(t))e^{-\lambda t}dt$$

$$v^\lambda(x,a) = \inf_{\alpha\in\mathcal{A}(a)} J^\lambda(x,\alpha)$$

Problem with long run average cost criterion

$$J(x,\alpha) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \ell(y_x(t,\alpha),\alpha(t))dt$$

$$\mu(x,a) = \inf_{\alpha\in\mathcal{A}(a)} J(x,\alpha)$$

The infinite horizon problem

Assumption: weak controllability property

There exist $C, \gamma > 0$ such that, for any $(x,a), (\bar{x}, \bar{a}) \in \overline{\Omega} \times [0,1]$ and any control $\alpha \in \mathcal{A}(a)$, there exist a control $\bar{\alpha} \in \mathcal{A}(\bar{a})$ and small times $\tau, \bar{\tau}$ with $0 \le \tau \le \bar{\tau} \le C\left(\|x-\bar{x}\| + |a-\bar{a}|\right)^{\gamma}$ which verifies

$$y_x(\tau, \alpha) = y_{\bar{x}}(\bar{\tau}, \bar{\alpha}) \qquad \text{and} \qquad \alpha(\tau) = \bar{\alpha}(\bar{\tau}).$$

Proposition

- For any $\lambda > 0$, the value function v^{λ} is continuous in $\overline{\Omega} \times [0,1]$, non increasing in the variable a, and $|v^{\lambda}(x,a)| \le M_{\ell}/\lambda$, where M_{ℓ} bounds ℓ.
- v^{λ} satisfies the Dynamic Programming Principle (DPP): for any $(x,a) \in \overline{\Omega} \times [0,1]$ and $T > 0$,

$$v^{\lambda}(x,a) = \inf_{\alpha \in \mathcal{A}(a)} \int_0^T \ell(y_x(t,\alpha), \alpha(t))\, e^{-\lambda t}\, dt + v^{\lambda}(y_x(T,\alpha), \alpha(T))\, e^{-\lambda T}$$

- Under the weak controllabilty property, the family $\{v^{\lambda}\}_{\lambda > 0}$ is equicontinuous in $\overline{\Omega} \times [0,1]$.

Constrained viscosity solutions

A continuous function $w : \overline{\Omega} \times [0,1] \to \mathbb{R}$ is a constrained viscosity solution of

$$F(x, a, w(x,a), Dw(x,a)) = 0 \qquad \text{in } \overline{\Omega} \times [0,1], \tag{3}$$

if w is a viscosity subsolution of (3) in $\Omega \times (0,1)$ and a constrained viscosity supersolution of (3) in $\overline{\Omega} \times [0,1]$, that is:

- $\forall \phi \in C^1(\mathbb{R}^N)$,
  $$F(x_0, a_0, w(x_0,a_0), D\phi(x_0,a_0)) \geq 0,$$
  at any local maximum point $(x_0, a_0) \in \Omega \times (0,1)$ of $w - \phi$.
- $\forall \phi \in C^1(\mathbb{R}^N)$,
  $$F(x_0, a_0, w(x_0,a_0), D\phi(x_0,a_0)) \leq 0,$$
  at any local minimum point $(x_0, a_0) \in \overline{\Omega} \times [0,1]$ of $w - \phi$.

HJB variational inequality

Theorem

The value function v^λ is a constrained viscosity solution in $\overline{\Omega} \times [0,1]$ of the HJB variational inequality

$$\min\left((Hw - \lambda w)(x,a), -\frac{\partial w(x,a)}{\partial a} \right) = 0, \qquad (4)$$

being

$$Hw = \frac{\partial w}{\partial x} f + \frac{\partial w}{\partial a} g + \ell.$$

Comparison Principle

Let v and w continuous functions in $\overline{\Omega} \times [0,1]$ which are respectively viscosity subsolution of (4) in $\Omega \times (0,1)$ and constrained viscosity supersolution of (4) in $\overline{\Omega} \times [0,1]$. Then $v \leq w$ in $\overline{\Omega} \times [0,1]$.

Corolary

The value function v^λ is the unique constrained viscosity solution in $\overline{\Omega} \times [0,1]$ of the HJB variational inequality (4).

Asymptotic behavior

Theorem

1. There exists $\mu \in \mathbb{R}$ such that

$$\lim_{\lambda \to 0+} \lambda v^{\lambda}(x,a) = \mu, \qquad \text{uniformly in } \overline{\Omega} \times [0,1]$$

and

$$\mu = \inf_{\alpha \in \mathcal{A}(a)} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \ell(y_x(t,\alpha), \alpha(t))dt.$$

2. For any $v \in \bigcap_{\varsigma > 0} \overline{\bigcup_{\varsigma > \lambda > 0} \{v^{\lambda}(\cdot,\cdot) - v^{\lambda}(x_0, a_0)\}}$,
(μ, u) is a constrained viscosity solution in $\Omega \times [0,1]$ of the HJB variational inequality:

$$\min\left((Hw - \mu)(x,a), -\frac{\partial w(x,a)}{\partial a} \right) = 0, \qquad (5)$$

with

$$Hw = \frac{\partial w}{\partial x} f + \frac{\partial w}{\partial a} g + \ell.$$

Outline

Linear finite elements

We approximate $\overline{\Omega}$ by a polyhedron in $\mathbb{R}^n$

$$\Omega^k = \bigcup_j S_j^k,$$

being S_j^k regular simplices with maximum diameter k, and we define $V^k = \{x^i\}_{i=1}^L$ as the set of nodes of Ω^k, with $L = \sharp(\Omega^k)$.
The interval [0,1] is discretized by an equidistant mesh

$$I_k = \left\{(j-1)\frac{1}{M-1} \ : \ j = 1, ..., M\right\}.$$

We consider the set W_k of functions $w : \Omega^k \times I_k \to \mathbb{R}$ with $w(\cdot, a)$ continuous in Ω^k and $\dfrac{\partial w(\cdot, a)}{\partial x}$ constant in the interior of each simplex of Ω^k.
Consequently $w \in W_k$ is fully determined by the values $w(x^i, a^j)$, $x^i \in V_k$, $a^j \in I_k$, so we can identify W_k with $\mathbb{R}^{LM}$.

Discretized problems

For an adequate parameter $h > 0$, we introduce the discretized problems:

$$\left| \begin{array}{l} P_k^\lambda\, w(x^i, a^j) = \min\left\{ D^\lambda w(x^i, a^j)\,,\; w(x^i, a^{j-1}) \right\}, \quad \text{if } a^j \neq 0, \\ \\ P_k^\lambda w(x^i, 0) = D^\lambda w\left(x^i, 0\right), \end{array} \right.$$

where $D^\lambda w(x^i, a^j) = h\ell(x^i, a^j) + (1 - \lambda h)\, w\left(x^i + h\, f(x^i, a^j), a^j + h\, g(a^j)\right)$.

Problem $\mathbf{P}_k^\lambda$: Find the fixed point of P_k^λ.

$$\left| \begin{array}{l} P_k(w, \mu)(x^i, a^j) = \min\left\{ Dw(x^i, a^j) - h\mu\,,\; w(x^i, a^{j-1}) \right\}, \quad \text{if } a^j \neq 0, \\ \\ P_k(w, \mu)(x^i, 0) = Dw\left(x^i, 0\right) - h\mu, \end{array} \right.$$

where $Dw(x^i, a^j) = h\ell(x^i, a^j) + w\left(x^i + h\, f(x^i, a^j), a^j + h\, g(a^j)\right)$.

Problem $\mathbf{P}_k$: Find (w_k, μ_k) such that $P_k(w_k, \mu_k) = w_k$.

Some results

Proposition

The operator $(P_k^\lambda)^M$ is contractive, so $\mathbf{P}_k^\lambda$ has a unique solution w_k^λ. Such solution verifies:

$$\left|w_k^\lambda(x^i, a^j)\right| \leq \frac{M_f}{\lambda}.$$

Proposition

There exists at most one $\mu \in \mathbb{R}$ such that problem $\mathbf{P}_k$ has a solution. Assuming that, given $(x^{i_0}, a^{j_0}) \in V_k \times I_k$, there exists $C > 0$ such that

$$\left|w_k^\lambda(x^i, a^j) - w_k^\lambda(x^{i_0}, a^{j_0})\right| \leq C,$$

then any (w, μ_k) verifying

1. $\lim_{\lambda \to 0+} \lambda w_k^\lambda(x^i, a^j) = \mu_k \qquad \forall x^i \in V_k, \ \forall a^j \in I_k.$
2. $w \in \bigcap_{\varsigma > 0} \left(\overline{\bigcup_{\varsigma > \lambda > 0} \left\{ w_k^\lambda - w_k^\lambda(x^{i_0}, a^{j_0}) \right\}} \right),$

is a solution of problem $\mathbf{P}_k$.

Drawbacks

The previous results suggest to approximate a solution of problem $\mathbf{P}_k$ by solving problem $\mathbf{P}_k^\lambda$ for λ near to zero with a fixed point algorithm, but this approach has several drawbacks.

- Solving problem $\mathbf{P}_k^\lambda$ by simple iteration is too slow, the contractive operator is not P_k^λ but its power $(P_k^\lambda)^M$, so as M increases is more and more expensive to perform a contraction.
- When a contraction is obtained, its linear parameter is of order $1 - h\lambda$, so its tends to one for small λ (and for small h).

Work in progress

- Consider policies with transition probabilities given by the barycentric coordinates of $.A : V_k \times I_k \to \{0, 1\}$.
- Associate to a policy A the Markov chain associated to the discretization where the states are the points (x^i, a^j) and
  - if $A(x^i, a^j) = 0$, there is a deterministic transition $T(x^i, a^j) = (x^i, a^{j-1})$.
  - if $A(x^i, a^j) = 1$, the transition probability to a point (x^p, a^q) is given by the barycentric coordinates $\beta_{i,j}^{p,q}$, where

$$(x^i + hf(x^i, a^j), a^j + hg(a^j)) = \sum_{p,q} \beta_{i,j}^{p,q}(x^p, a^q).$$

- Construct a policy iteration algorithm.
- Obtain estimates for the approximations.

WORKSHOP ANNOUNCEMENT

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