On Neumann problems for nonlocal Hamilton-Jacobi equations related to jump processes

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Outline

Neumann boundary value problems of the following kind:

$$\begin{cases} u(x) - \mathcal{I}[u](x) + H(x, Du) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

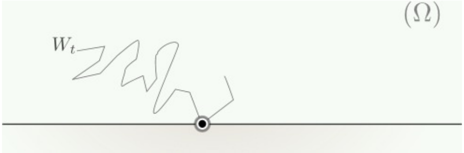
- the diffusion $\mathcal{I}$ is a nonlocal operator, in our context a singular integral term,
- H is a nonlinear Hamiltonian,
- $\Omega \subset \mathbb{R}^n$ is a general domain (enough smooth).

Results:

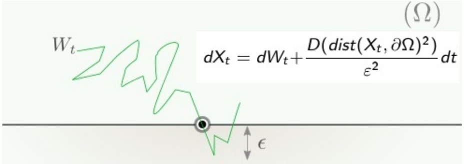
- Existence and uniqueness for the stationary problem.
- Applications to evolutive problem: existence, uniqueness and asymptotic behaviour as $t \to +\infty$ of the evolutive problem.

Reference: *D. Ghilli, On Neumann problems for nonlocal Hamilton-Jacobi equations with dominating gradient terms (2016), Sumbitted, arXiv: 1601.04308*

Probabilistic approach



By a reflection on the boundary- *Lions-Snitzman penalization procedure*:



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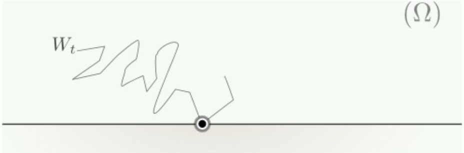

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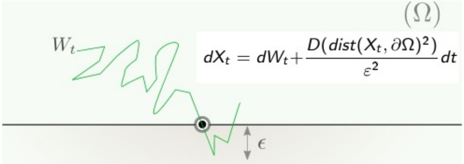
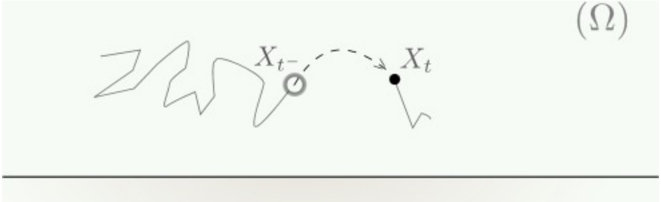


KEY RESULT: For a PDE with Neumann or oblique boundary conditions, there is a unique underlying reflection process. (Lions-Snitzman, Barles-Lions).

REMARK: **This relies on the underlying stochastic processes being continuous (at least in the case of normal reflections)**.

More general: Lévy processes

Càdlàg: right continuous limits on the left



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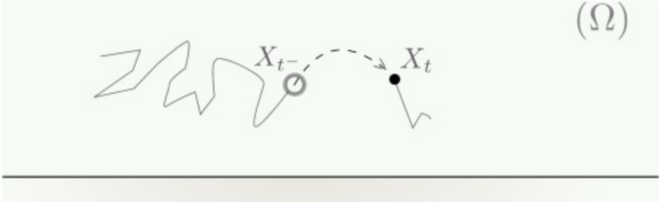


For example, used in mathematical finance to model the behaviour of stochastic volatility

- Ornstein-Uhlenbeck with a Levy diffusion: Barndorff-Nielsen and Shephard 01
- option pricing: Nicolato-Venerdos 03, Hubalek-Sgarra 09, 11,
- portfolio optimisation: Benth-Karlsen-Reikvam 03,
- multiple scales and stochastic jump volatility: Bardi-Cesaroni-Scotti 16.

Infinitesimal generators of Lévy processes

Remark

Analytical approach: study the infinitesimal generator of the stochastic process.

Nonlocal operator

$$\begin{aligned} \mathcal{I}[u](x) &= P.V. \int_{\mathbb{R}^n} [u(x+z) - u(x)] d\mu(z) \\ &= \lim_{\delta \to 0^+} \int_{|z|>\delta} [u(x+z) - u(x)] d\mu(z) \end{aligned}$$

where μ is a singular positive Borel measure satisfying some integrability condition and representing the intensity of the jump from x to $x+z$

Lévy measures

General integrability condition at $z = 0$ and at infinity satisfied by Lévy measures:

$$\int (|z|^2 \wedge 1) d\mu(z) < \infty.$$

We mainly consider

$$\frac{d\mu(z)}{dz} \sim \frac{1}{|z|^{n+\sigma}} \quad \sigma \in (0,2).$$

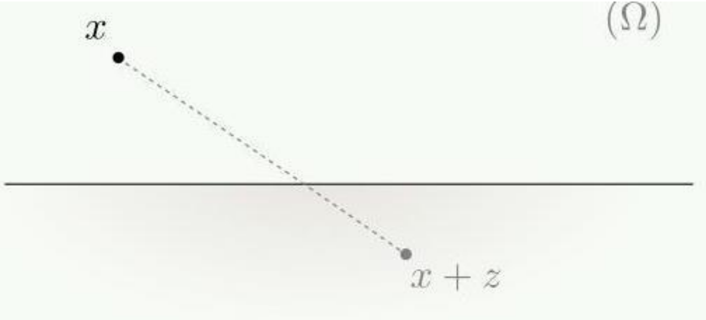
Stable, tempered stable, and the larger class of self-decomposable processes in $\mathbb{R}^n$.

If $\sigma \in (0,1)$, then

- $\int (|z| \wedge 1) d\mu(z) < \infty$
- $\mathcal{I}[\phi]$ with ϕ bounded and C^1 is well-defined

Neumann boundary condition

In the case of a jump process, there are several ways to keep the process inside the domain. For example.. (take $\Omega =$"halfspace")



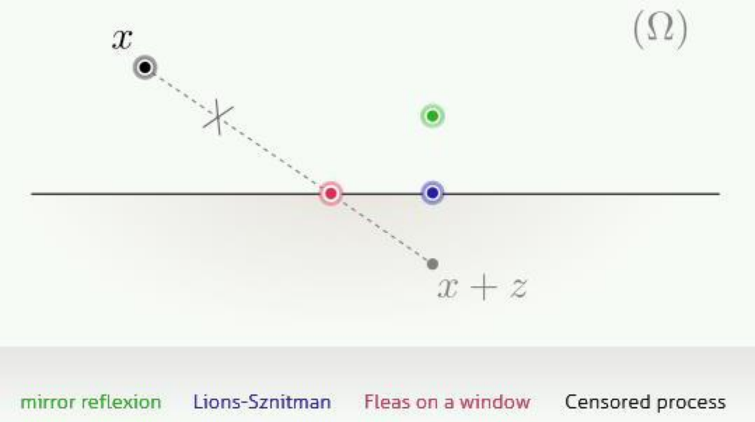
Linear PIDE-BCGJ

G. Barles, E.Chasseigne, C. Georgeline, E. R. Jacobsen, 2014, "On neumann type problems for non-local equations set in a half space"

- linear equations: $u(x) - \mathcal{I}[u](x) + f(x) = 0$ + Neumann BC
- domains with flat boundary

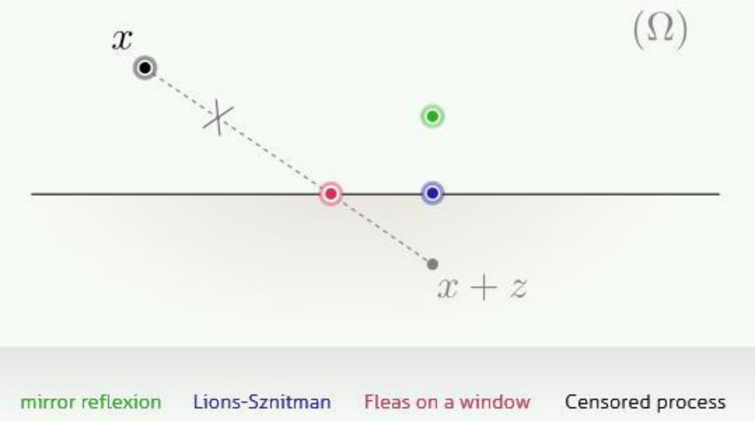
(At least) four different and coherent (w.r.t the classical case) models of reflection.

Models of reflections



Don't jump outside, censored

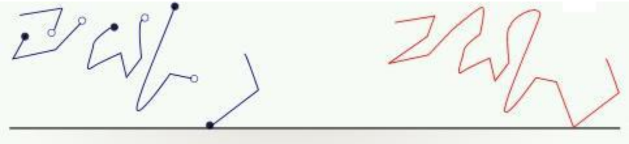
We focus on the so-called censored process



Any outwards jump is cancelled (censored) and the process is restarted (resurrected) at the origin of that jump.

Good models?

$$d\mu^\sigma = c_\sigma |z|^{-(N+\sigma)} dz \quad \underset{\sigma \to 2^-}{\longrightarrow} \quad \begin{array}{c} u - \triangle = f \\ \frac{\partial u}{\partial n} = 0 \end{array}$$



Each model is good!

The censored process/operator

Actually we don't jump outside:

$$\mathcal{I}[u](x) = P.V. \int_{x+z \in \bar{\Omega}} [u(x+z)) - u(x)] d\mu(z)$$

- The Neumann condition influences the equation inside the domain.
- No need for conditions on Ω^c.
- Note the bad dependence on x in the domain of integration

Underlying process-probabilistic references

In the censored case, the process can be constructed via some probabilistic methods:

Refs:K. Bogdan, K. Burdzy and Z. Chen. 2003
Q.Y. Guan. and Z.M. Ma. 2006
M. Fukushima, Y. Oshima and M. Takeda. 1994
N. Jacob 2005

Related to the censored stable processes of Bogdan et al. and the reflected σ-stable process of Guan and Ma.

Such processes can be constructed from a Brownian motion by first subordinating the process and then reflecting it (or also, by reflecting the Brownian motion first and then subordinate the reflected process.)

Type of singularity

We consider

$$\frac{d\mu(z)}{dz} \sim \frac{1}{|z|^{n+\sigma}}, \quad \sigma \in (0,1).$$

Why $\sigma < 1$?

Stronger singuarity, $\sigma \in [1,2)$?

Interesting (but difficult) case.
For linear equations by Barles,Chasseigne,Georgeline, Jacobsen: comparison in some Hölder continuity class of functions.
Even in the case of linear equations the results could not be optimal.

$\sigma < 1$, not standard

- nonlinear term (even in domains with flat boundary, see next slides);
- general domain.

$\sigma < 1$ in the halfspace

Not standard due to the Hamiltonian term! Why?

Preliminary remark

Since $\int (1 \wedge |z|) \frac{1}{|z|^{n+\sigma}} < +\infty$, then

$$\mathcal{I}[u](x) = \int_{x+z \in \bar{\Omega}} [u(x+z) - u(x)] \frac{dz}{|z|^{n+\sigma}} dz$$

in integrable when $|z| \geq 1$, that is enough far from zero.

"The main difficulties are near the boundary."

$\sigma < 1$ in the halfspace

LINEAR EQUATIONS $\sigma < 1$, framework considered by BCGJ:

$$u(x) - \mathcal{I}[u](x) + f(x) = 0 \text{ in } \Omega + \quad \text{Neumann BC}$$

The process does not reach the boundary!.
That is, we use the existence of a blow-up function on the boundary which allows to stay far from the boundary.

NONLINEAR EQUATIONS $\sigma < 1$, our framework:

$$u(x) - \mathcal{I}[u](x) + H(x, Du) = 0 \text{ in } \Omega + \quad \text{Neumann BC}$$

In general, the blow-up function method does not work.
Roughly speaking, the nonlinear terms could push the process to hit the boundary.
Further difficulties.

Assumption on the Hamiltonian

IMPORTANT:

the growth of H in the gradient strictly dominates the nonlocal diffusion.

Two types of H:

- either coercive in the gradient term with superfractional growth

$$H(x,p) = b(x)|p|^m + a_1(x)|p|^l + (a_2(x), p) - f(x),$$

  where $m > \sigma$, $b(x) \geq b_0 > 0$;

- either of Bellman type, not necessarely coercive

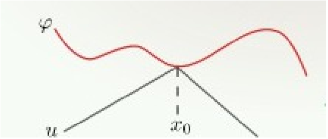
$$H(x,p) = \sup_{\alpha \in \mathcal{A}} \{-b(x,\alpha) \cdot p - l(x,\alpha)\}$$

  where $\mathcal{A}$ is a compact metric space, $b : \bar{\Omega} \times \mathcal{A} \to \mathbb{R}^n$ and $f : \bar{\Omega} \times \mathcal{A} \to \mathbb{R}$ are continuous and bounded functions.

  (i) Uniform continuity of the cost l;

  (ii) Uniform Lipschitz continuity of the drift b.

Viscosity solutions



- The solutions in general are not regular, in this way we give a sense to the differential terms.
- Useful to deal with the integral terms when the measure is singular, by replacing u with a regular function: $\int_{\mathbb{R}^n}[u(x+z)-u(x)]d\mu(z)$
- Useful definition of "generalized" boundary conditions

Main results

Recall the Neumann boundary value problem:

$$\begin{cases} u(x) - \mathcal{I}[u](x) + H(x, Du) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

COMPARISON [G. 2016]

Under the assumptions of the previous slides, let u be a bounded usc subsolution of (1) and v a bounded lsc supersolution of (1). Then

$$u \leq v \text{ in } \bar{\Omega}.$$

EXISTENCE, UNIQUENESS

By Perron's method for integro-differential equations, in the class of continuous functions.

ASYMPTOTIC BEHAVIOUR FOR THE EVOLUTIVE PROBLEM [G. 2016]

Convergence as $t \to +\infty$ of the solutions of the evolutive problem to the associated stationary problem.

Perspectives and open questions

- More general Neumann condition: inhomogeneous and linear.
- Other models of reflection, mirror and fleas (normal projection done by Barles Georgeline Jacobsen). First question: how to define the reflection in a general domain?
- Case $\sigma \in [1, 2)$. Very difficult.. First step: consider solutions which are in some sense Hölder continuous up to the boundary and establish the comparison in this class.

Perspectives and open questions

- More general Neumann condition: inhomogeneous and linear.
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Muito obrigado pela sua atenção!

Further references

Menaldi, Robin, 1985: reflection problems solved in the case of diffusion processes with jumps only inside Ω.
Garroni, Menaldi, 2002: large class of uniformly elliptic integro-differential equations, where the principal part is a non-degenerate 2nd order term. Dirichlet type problems.

Natural Neumann boundary condition

Guan and Ma: the boundary condition arising through the variational formulation and Green type formulas is

$$\lim_{t\to 0} t^{2-\sigma}\frac{\partial u}{\partial x_n}(x+te_n)=0.$$

This formula allows the normal derivative $\frac{\partial u}{\partial x_n}$ to explode less rapidly than $|x_n|^{\sigma-2}$ and justifies the use of boundary conditions in the viscosity sense since u_{x_n} is not necessarily equal to 0 on the boundary for $\sigma < 2$.

Financial models with stochastic jump volatility

Financial stochastic volatility models of the type:

$$dX_s = \gamma(X_s, Y_s)ds + \sigma(X_s, Y_s)dW_s$$

Non-Gaussian, jump model

$$dY_s = -Y_s ds + \tau dZ_s$$

where Z_s is a pure jump Levy process with positive increments.
The non-Gaussian model was used for option pricing: Nicolato - Venerdos 03, Hubalek - Sgarra 09, 11
and for portfolio optimisation: Benth - Karlsen - Reikvam 03.

For example: fast stochastic volatility, multiscale financial models.
Singular perturbation of nonlocal HJ equations. Large deviations of fast stochastic volatility models.

Evolutive problem, large time behaviour

We provide large time behaviour for the problem

$$\begin{cases} \partial_t u(x) - \mathcal{I}[u(\cdot,t)](x) + H(x,Du) = 0 & \text{in } \Omega \times (0,+\infty) \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \bar{\Omega}. \end{cases} \tag{2}$$

- Ω is a bounded open subset of $\mathbb{R}^n$ of class $W^{2,\infty}$
- H is an Hamiltonian in coercive form
- $$\mathcal{I}[u(\cdot,t)](x) = P.V. \int_{x+z\in\bar{\Omega}} [u(x+z,t) - u(x,t)] \frac{dz}{|z|^{N+\sigma}}$$

  with $\sigma \in (0,1)$.

Large time behaviour

Let $u(x,t)$ be the unique solution to the previous evolution problem.

Question: What is the behaviour of $u(x,t)$ as $t \to +\infty$?
Does it converge to a/the solution of a suitably associated problem? And how?

Ergodic behaviour

- $$u(\cdot,t)/t \to c$$
  where c is the so-called ergodic constant.
- We search for an asymptotic development of u of the type
  $$u(x,t) = ct + w(x) + o(1)$$
  where $o(1)$ tends to zero as $t \to \infty$.

Large time behaviour

(i) Prove that there exists a unique constant $c \in \mathbb{R}$ for which the stationary ergodic problem

$$\begin{cases} -\mathcal{I}[w(\cdot)](x) + H(x, Dw) = -c & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

has a solution $w \in C^{\frac{m-\sigma}{m}}(\bar{\Omega})$ (unique up to an additive constant).

(ii) Convergence as $t \to +\infty$:
There exists a pair (w, c) solution to (3) such that

$$u(x,t) - ct - w(x) \to 0 \text{ as } t \to +\infty$$

uniformly on $\bar{\Omega}$.

Main steps

We follow closely the arguments given in G. Barles, O. Ley, S. Koike, E. Topp, *Regularity results and large time behavior for integro-differential equations with coercive hamiltonians* (see also G. Barles, E. Chasseigne, A. Ciomaga, C. Imbert, *Lipschitz regularity of solutions for mixed integro-differential equations* in the local framework).

- Key result: Hölder regularity for the solutions of (2): G. Barles, O. Ley, S. Koike and E. Topp (see above)
- We solve the ergodic problem, by means of the approximant problems and uniform estimates given by regularity results.
- We prove a strong maximum principle for the evolutive problem considered (2).
- We conclude the asymptotic convergence as $t \to +\infty$.