# Distributed Algorithms for Aggregative Games on Graphs

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# Outline

# An Aggregative Game

We have a system with $N$ agents, where each agent $i$ solves the following problem:

$$\begin{aligned} &\text{minimize} && f_i(x_i, s(x)), \quad s(x) = \sum_{j=1}^{N} x_j \\ &\text{subject to} && x_i \in K_i \subseteq \mathbb{R}^n. \end{aligned} \tag{AggGame}$$

- $f_i(\cdot,\cdot)$ and $K_i$ are known only to agent $i$
- Agent $i$ decides on $x_i$, and does not have access to the decisions $x_j$ of the other agents
- Unlike [Jensen06, MartimortStole2010], the emphasis here is on computation of an equilibrium in the absence of instantaneous access to the aggregate value $\bar{x}$
- This can be achieved when the agents are able to communicate locally over a connected network

# Motivating Example

## Example (Networked Nash-Cournot Oligopoly)

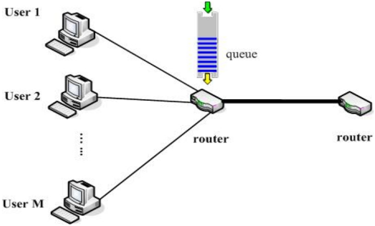
- $N$ firms competing at $M$ locations
- Optimization problem for firm $i$

$$\begin{aligned}
\text{minimize} \quad & \sum_{\ell=1}^{M} \left(c_{i\ell}(x_{i\ell}) - p_\ell(\bar{s}_\ell)s_{i\ell}\right) + t_i(x_{i1}, \ldots, x_{iM}) \\
\text{subject to} \quad & \sum_{\ell=1}^{M} x_{i\ell} = \sum_{\ell=1}^{M} s_{i\ell}, \\
& x_{i\ell}, s_{i\ell} \geq 0, \quad x_{i\ell} \leq \mathrm{cap}_{i\ell}, \qquad \ell = 1, \ldots, M.
\end{aligned}$$

- $x_{i\ell}$ and $s_{i\ell}$ are production and sales for firm $i$ at location $\ell$
- $p_\ell$ denotes the price function and $\bar{s}_\ell = \sum_{j=1}^{N} s_{j\ell}$
- $p_\ell(\bar{s}_\ell)s_{i\ell}$ is the revenue at location $\ell$ for firm $i$
- $t_i$ is the transportation cost

# Flow control games

- Multitude of applications on the internet
- A large number of users - inherently noncooperative in nature




- Demands for bandwidth lead to congestion
- Solution concept - a Nash equilibrium in flow rates, where each agent maximizes its utility less cost of sending flow

- User $i$ has utility given by $U_i(x_i) := \ln(1 + x_i) + d_i$, $x_i \geq 0$,
- Cost of sending flow can be modeled by $P_i(x_i, x_{-i}) := k_i x_i^2 + Mp(C - \sum_{i=1}^{N} x_i)$

**Nash Equilibrium in flow decisions** $x^* = (x_1^*, \ldots, x_N^*)$

- Users maximize the benefit of sending flow by solving

$$\text{Player } i \qquad \begin{array}{ll} \text{maximize} & U_i(x_i) - P_i(x_i; x_{-i}) \\ \text{subject to} & x_i \geq 0, \end{array}$$

- Nash equilibrium in flow generation levels: $(x_1^*, \ldots, x_N^*)$

# Computational algorithm

$$\text{minimize} \quad f_i(x_i, s(x)), \qquad s(x) = \sum_{j=1}^{N} x_j$$

$$\text{subject to} \quad x_i \in K_i \subseteq \mathbb{R}^n. \qquad \text{(AggGame)}$$

- Are there "distributed" algorithms? [1]
- Under ideal conditions and suitable assumption, each player updates:

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha \nabla_{x_i} f_i(x_i^k,\ s(x^k)\ )]$$

- What is the challenge then?
  - No agent knows $s(x^k) = \sum_{i=1}^{N} x_i^k$
  - No central entity exists to provide it.

[1] Projection based [Facchinei-Pang03, Alpcan03] and their regularized variants [Kannan-Shanbhag10]

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# The central theme

Goal :

- Develop distributed algorithms that can aid agents to learn the aggregate and lead to an equilibrium point
- And still satisfy the informational constraints

How?

- By allowing agents to communicate their beliefs of the aggregate: $(\sum_{i=1}^{N} x_i)$
- But not their decisions: $(x_i)$
- This is in sharp contrast with "distributed optimization" setting (since here agents are competing)

# Assumptions for a Unique Equilibrium

**Basic Assumption 1**

For each $i = 1, \ldots, N$,

- The set $K_i \subset \mathbb{R}^n$ is compact and convex
- Function $f_i(x_i, y)$ is continuously differentiable in $(x_i, y)$ over some open set containing the set $K_i \times \bar{K}$, where $\bar{K} = \sum_{i=1}^{N} K_i$
- The function $x_i \mapsto f_i(x_i, s(x))$, $s(x) = \sum_{j=1}^{N} x_j$, is convex over the set $K_i$.

- These assumptions are sufficient for the existence of an equilibrium.

# Additional Assumption: Uniqueness

- Define mapping $\Phi(x)$ by

$$\Phi(x) \triangleq \begin{pmatrix} F_1(x_1, s(x)) \\ \vdots \\ F_N(x_N, s(x)) \end{pmatrix}$$

$$F_i(x_i, s(x)) = \nabla_{x_i} f_i(x_i, s(x)), \qquad s(x) = \sum_{j=1}^{N} x_j$$

**Basic Assumption 2 [Strict Monotonicity]**

The mapping $\Phi(x)$ is strictly monotone over $K_1 \times \cdots \times K_N$, i.e.,

$$(\Phi(x) - \Phi(x'))^T (x - x') > 0, \quad \forall x, x' \in K_1 \times \cdots \times K_N.$$

# Equilibrium conditions

**Unique Equilibrium**

Consider the aggregative Nash game defined in (AggGame). Suppose Basic Assumptions hold. Then, the game admits a unique Nash equilibrium.

# Further Assumptions

**Basic Assumption 3 [Computational]**

The mapping $F_i(x_i, u)$ is uniformly Lipschitz continuous in $u$ over $\bar{K} = K_1 + \cdots + K_N$, for every fixed $x_i \in K_i$ i.e., for some $L_{-i} > 0$

$$\|F_i(x_i, u) - F_i(x_i, z)\| \leq L_{-i}\|u - z\|.$$

# Outline

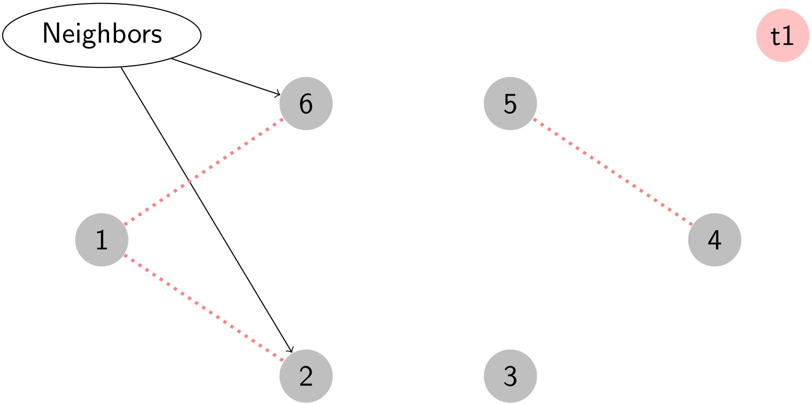
# Synchronous setting

- A time varying undirected connectivity graph $\mathcal{G}_k = (\mathcal{N}, \mathcal{E}_k)$
- $\mathcal{N} = \{1, \ldots, N\}$ be a set of $N$ agents
- $\mathcal{E}_k$ is the set of edges at time $k$
- $\mathcal{N}_i(k)$ : immediate neighbors of agent $i$ at time $k$

## Connectivity Assumption

There exists an integer $Q \geq 1$ such that the graph $\bigcup_{l=1}^{Q} \mathcal{G}_{l+k}$ is connected for all $k$.

# Limited connectivity of agents



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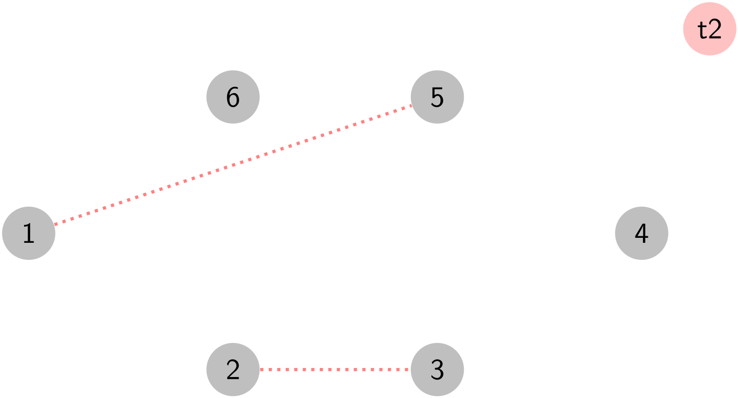

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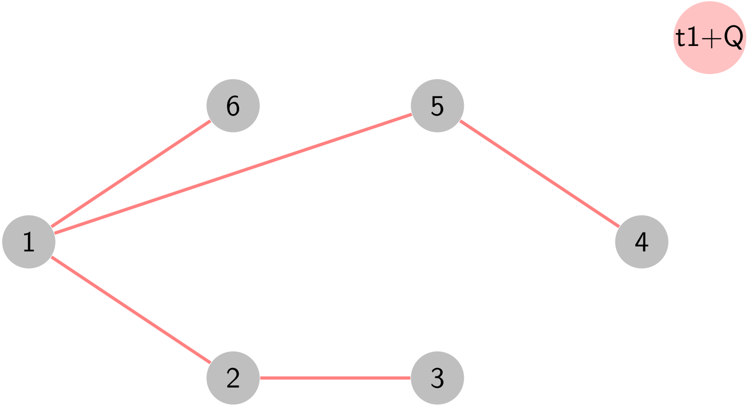

# Variational Inequality Formulation

We want to determine the point $x^* = (x_1^*, \ldots, x_N^*) \in K$ such that

$$\langle \Phi(x^*), x - x^* \rangle \geq 0 \qquad \text{for all } x \in K,$$

where

$$\Phi(x) \triangleq \begin{pmatrix} F_1(x_1, s(x)) \\ \vdots \\ F_N(x_N, s(x)) \end{pmatrix}$$

$$F_i(x_i, s(x)) = \nabla_{x_i} f_i(x_i, s(x)), \qquad s(x) = \sum_{j=1}^{N} x_j$$

We use a suitable analog of

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha \nabla_{x_i} f_i(x_i^k, s(x^k))$$

where we distribute learning of $s(x^k)$ in the network

# Outline of the synchronous algorithm: the three steps

1. Learn aggregate:[2]
   - Each player maintains and updates an estimate $v_i^k$ of $s(x^k) = \sum_{i=1}^{N} x_i^k$ (tricky since $x_j^k$'s are players' private variables & are changing in time)
   - Every player communicates its estimate $v_i^k$ to the neighbors
2. Update decision:
   - Using aligned estimate agents update their decision ($x_i^k$)
3. Update estimate of aggregate:
   - Agents update their estimate to reflect their current decision.

[2] Inspiration: Consensus based algorithm [?, ?], Distributed optimization problem [?, ?]

# Synchronous algorithm

1. Learn aggregate:

$$\hat{v}_i^k = \sum_{j\in\mathcal{N}_i(k)} w_{ij}(k) v_j^k, \quad v_i^0 = x_i^0,$$

$w_{ij}(k)$ : agent $i$'s weight to agent $j$'s information at step $k$

2. Update decision:

$$x_i^{k+1} = \Pi_{K_i}[x_i^k - \alpha_k F_i(x_i^k, \boxed{N\hat{v}_i^k})].$$

where agent $i$ uses its estimate $N\hat{v}_i^k$ instead of $\sum_{j=1}^N x_j^k$

3. Update estimate of aggregate:

$$v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k$$

Step suggested by Ram in [Ram-Nedić-Veeravali 2012]

# Weight assumptions

## Synchronous Weight

For all $i \in \mathcal{N}$ and all $k \geq 0$, the following hold:

(i) $w_{ij}(k) \geq \delta$ for all $j \in \mathcal{N}_i(k)$ and $w_{ij}(k) = 0$ for $j \notin \mathcal{N}_i(k)$;

(ii) $\sum_{j=1}^{N} w_{ij}(k) = 1$ for all $i$;

(iii) $\sum_{i=1}^{N} w_{ij}(k) = 1$ for all $j$.

# Stepsize assumptions

## Synchronous Stepsize

The stepsize $\alpha_k$ is chosen such that the following hold:

(i) The sequence $\{\alpha_k\}$ is monotonically non-increasing i.e., $\alpha_{k+1} \leq \alpha_k$ for all $k$;

(ii) $\sum_{k=0}^{\infty} \alpha_k = \infty$;

(iii) $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$.

# Convergence of synchronous algorithm

**Proposition (Koshal-AN-Shanbhag 2016)**

*Let Basic Assumptions hold. Further let the Synchronous Weight and Synchronous Stepsize assumption hold. Then, the sequence $\{x^k\}$ generated by the Synchronous Algorithm converges to the (unique) solution $x^*$ of the game.*

# Proof sketch

1. Gradient projection algorithm technique $(\|x^k - x^*\|)$
    - Utilizing the Lipschitz continuity of the map leads to a bound on $\|x_i^{k+1} - x_i^*\|$ in terms of $\|Nv_i^k - \bar{x}^*\|$
2. Estimate the errors for the beliefs $\|Nv_i^k - \sum_{i=1}^N x_i^*\|$
    - Introduce $y^k = \sum_{\ell=1} v_\ell^k / N$, the average estimate under perfect information
    - $y^k$ also tracks the average of the aggregate i.e., $y^k = \sum_{\ell=1}^N x_\ell^k / N$
    - Construct bounds for $\|v_i^k - y^k\|$ and $\|Ny^k - \sum_{i=1}^N x_i^*\|$

# Moving ahead

Convergence established for synchronous regime but

Key Drawback:

- Synchronization is a challenge in large networks
- Requires coordination in terms of stepsize

Remedy: Desynchronization

- No explicit dependency
- Allows independent solution of stepsizes
- Drawback: cannot accommodate time-varying graphs

# Outline

# Asynchronous setting

## Connected Graph

The undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is connected.

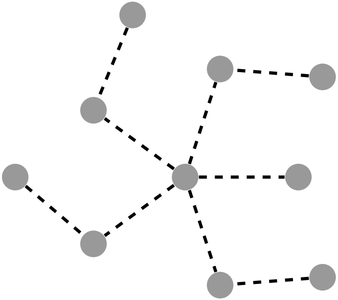

Figure: A depiction of a gossip communication.

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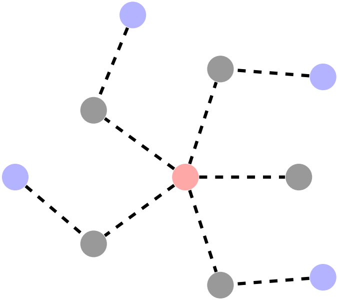

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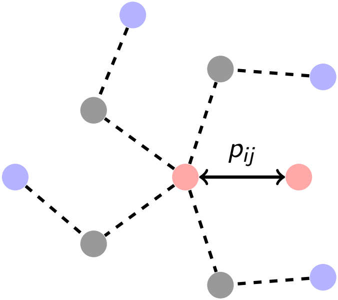


Figure: A depiction of a gossip communication.

# Outline of the asynchronous algorithm: the four steps

1. Agents have local clocks ticking at rate 1
2. Wake Up:
   - Agent $I^k$ wakes up at time $k$ and then contacts $J^k$
3. Learn aggregate:
   - Only agents $\{I^k, J^k\}$ communicate their estimate $(v_i^k)$ and perform mixing
4. Update decision:
   - Using aligned estimate agents update their decision $(x_i^k)$
5. Update estimate of aggregate:
   - Agents update their estimate to reflect their current decision.

# Asynchronous algorithm

- Wake Up:
- Agent $I^k$ wakes up at time $k$ and then contacts $J^k$

1. Learn aggregate:

$$\hat{v}_i^k = \frac{v_{I^k}^k + v_{J^k}^k}{2} \quad \text{for } i \in \{I^k, J^k\}, \quad v_i^0 = x_i^0,$$

2. Update decision:

$$x_i^{k+1} = \left(\Pi_{K_i}[x_i^k - \alpha_{k,i} F_i(x_i^k, N\hat{v}_i^k)] - x_i^k\right) \mathbb{1}_{\{i \in \{I^k, J^k\}\}} + x_i^k$$

3. Update estimate of aggregate:

$$v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k.$$

# Stepsize Assumptions

## Asynchronous Stepsize

The stepsize $\alpha_{k,i}$ is updated as follows:

$$\alpha_{k,i} = \frac{1}{\Gamma_k(i)}, \quad \text{where}$$

- $\Gamma_k(i)$ denotes the number of updates that agent $i$ has executed up to time $k$ inclusively.

# Convergence of asynchronous algorithm

**Proposition (Koshal-AN-Shanbhag 2016)**

*Let Basic Assumption, Connected Graph and the Asynchronous Stepsize assumption hold. Then, the sequence $\{x^k\}$ generated by the Asynchronous Algorithm converges to the (unique) solution $x^*$ of the game with probability 1.*

# Convergence analysis of asynchronous algorithm

Key property: almost surely we have for all $k$ large enough

$$\alpha_{k,i} \leq \frac{2}{kp_i}, \qquad \left|\alpha_{k,i} - \frac{1}{kp_i}\right| \leq \frac{2}{k^{3/2-q} p_{\min}^2}.$$

where $p_i$ is the probability that agent $i$ updates, $p_{\min} = \min_i p_i$ and $q \in (0, 1/2)$.

1. The stepsize
$$\alpha_{k,i} = \frac{1}{\Gamma_k(i)}$$
behaves almost surely as $1/k$ for all agents $i$ and for all $k$ large enough
2. Stochastic gradient technique $\left(\mathbb{E}\left[\|x^k - x^*\|^2 \mid \mathcal{F}_{k-1}\right]\right)$
3. Estimate the expected errors for the beliefs $\left(\mathbb{E}\left[\|Nv_i^k - \bar{x}^*\| \mid \mathcal{F}_{k-1}\right]\right)$

# Constant stepsize asynchronous algorithm

Agents employ a constant deterministic yet uncoordinated stepsize

- Wake Up:
- Agent $I^k$ wakes up at time $k$ and then contacts $J^k$

1. Learn aggregate:

$$\hat{v}_i^k = \frac{v_{I^k}^k + v_{J^k}^k}{2} \quad \text{for } i \in \{I^k, J^k\}, \quad v_i^0 = x_i^0,$$

2. Update decision:

$$x_i^{k+1} = \left(\Pi_{K_i}[x_i^k - \alpha F_i(x_i^k, N\hat{v}_i^k)] - x_i^k\right) \mathbb{1}_{\{i \in \{I^k, J^k\}\}} + x_i^k$$

3. Update estimate of aggregate:

$$v_i^{k+1} = \hat{v}_i^k + x_i^{k+1} - x_i^k.$$

### Basic Assumption 4

The mapping $F_i(x_i, u)$ is uniformly Lipschitz continuous in $x_i$ over $K_i$, for every fixed $u \in \bar{K}$ i.e., for some $L_i > 0$

$$\|F_i(x_i, u) - F_i(y_i, u)\| \leq L_i \|x_i - y_i\|;$$

# Error bound: constant stepsize asynchronous algorithm

## Proposition

*Let Basic Assumptions hold with the mapping $\Phi$ be* strongly monotone *over the set $K$. Further let Connected Graph* assumption *hold. Then, the following holds for the sequence $\{x^k\}$ generated by the Asynchronous Algorithm with stepsize $\alpha_{k,i} = \alpha_i$*

$$\limsup_{k\to\infty} \mathbb{E}[\|x^{k+1} - x^*\|^2] \leq Err(N, n, \mathcal{G}),$$

*where $x^*$ is the unique equilibrium of the game.*

# Expression for error

$$Err(N,n,\mathcal{G}) = \frac{4p_{\max}\alpha_{\max}^2 C^2 N + 2p_{\max}\alpha_{\max}^2 B\frac{\sqrt{2nN}\,C}{1-\sqrt{\lambda}}}{2\mu p_{\min}\alpha_{\min} - 2p_{\max}(\max_i L_i)(\alpha_{\max}-\alpha_{\min})}$$

- where $\mu$ is the strong monotonicity constant of the gradient map $\Phi$ and $C$ is a bound on $\|\Phi_i(x^k)\|$
- $\alpha_{\max} = \max_{i=1,\ldots,N}\{\alpha_i\}$, $\alpha_{\min} = \min_{i=1,\ldots,N}\{\alpha_i\}$,
- $p_{\max} = \max_{i=1,\ldots,N}\{p_i\}$ and $p_{\min} = \min_{i=1,\ldots,N}\{p_i\}$.
- $p_i$ is the probability of agent $i$ updating
- $B = (\max_i L_{-i})NM$
- $M \geq \max_{x_i,z_i\in K_i}\|x_i - z_i\|$ for all $i$

# Expression for error - equal step sizes

$$Err(N,n,\mathcal{G}) = \frac{\alpha p_{\max}}{\mu p_{\min}}\left(2C^2N + B\frac{\sqrt{2nN}\,C}{1-\sqrt{\lambda}}\right)$$

- where $\alpha_{\max} = \max_{i=1,\ldots,N}\{\alpha_i\}$, $\alpha_{\min} = \min_{i=1,\ldots,N}\{\alpha_i\}$,
- $p_{\max} = \max_{i=1,\ldots,N}\{p_i\}$ and $p_{\min} = \min_{i=1,\ldots,N}\{p_i\}$.
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# Outline

# A Nash-Cournot game: a classic aggregative game

Firm $i$ solves the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & c_i(x_i) - x_i g(\bar{x}) \\ \text{subject to} \quad & x_i \in K_i, \end{aligned}$$

- $c_i(x_i) = u_i x_i + v_i x_i^2$
- $g(\bar{x}) = 100 - \bar{x}$
- $K_i = [0, 500]$
- Nash equilibrium exists and is unique

$$x_i^* = \frac{100 - u_i}{2v_i + 1} - \frac{1}{(2v_i + 1)} \frac{\sum_{j=1}^N \frac{100 - u_j}{(2v_j + 1)}}{\left(1 + \sum_{j=1}^N \frac{1}{(2v_j + 1)}\right)}$$

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# Simulation setup

- $u_i$ and $v_i$ are once chosen and remain fixed

$$u_i \sim U(2,10) \quad v_i \sim U(2,4) \quad \forall i$$

- Network size: $N = 20$, $50$, and $100$
- Terminate after $\tilde{k} = 5e3$ and $1e4$ iterations
- Report mean error and confidence interval for 50 sample paths

$$\mathrm{err}_\ell := \left\| x_\ell^{\tilde{k}} - x^* \right\|_\infty = \max_{1 \le i \le n} \{ |[x_\ell^{\tilde{k}}]_i - [x^*]_i| \}.$$

# Synchronous algorithm: Setting

- Generate an adjacency matrix $A(k)$ such that $\mathcal{G}$ is connected
- Given $A(k)$ generate the weight matrix $W(k)$ as

$$[W]_{ij} = \begin{cases} 0 & \text{if } A_{ij} = 0 \\ \delta & \text{if } A_{ij} = 1 \\ 1 - \delta d(i) & \text{if } i = j, \end{cases}$$

$d(i)$ : number of neighbors of player $i$, and

$$\delta = \frac{0.5}{\max_i\{d(i)\}}.$$

- Stepsize update rule

$$\alpha_{k,i} = \frac{\alpha}{k}, \quad \text{for all } i = 1, \ldots, N, \quad \text{where } \alpha = 0.5.$$

# Synchronous algorithm: Mean error

Table: Dynamic network

| $N$ | $\tilde{k}$= 5e3 | $\tilde{k}$= 1e4 |
|---|---|---|
| 20 | 6.49$e$-5 | 6.03$e$-6 |
| 50 | 1.13$e$-2 | 1.70$e$-3 |
| 100 | 2.29$e$-1 | 3.18$e$-2 |

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| $N$ | $\tilde{k}$= 5e3 | $\tilde{k}$= 1e4 |
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| 50 | 1.13$e$-2 | 1.70$e$-3 |
| 100 | 2.29$e$-1 | 3.18$e$-2 |

Table: Complete network

| $N$ | $\tilde{k}$= 5e3 | $\tilde{k}$= 1e4 |
|---|---|---|
| 20 | 1.11$e$-8 | 1.47$e$-9 |
| 50 | 5.07$e$-7 | 8.06$e$-8 |
| 100 | 3.26$e$-6 | 5.04$e$-7 |

# Synchronous algorithm: Confidence interval width

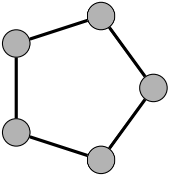
Table: Dynamic network

| $N$ | $\tilde{k}$= 5e3 | $\tilde{k}$= 1e4 |
|---|---|---|
| 20 | 5.91$e$-9 | 5.81$e$-10 |
| 50 | 5.03$e$-7 | 9.77$e$-8 |
| 100 | 1.32$e$-5 | 1.43$e$-6 |

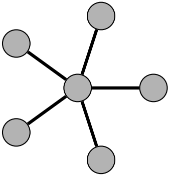
Table: Complete network

| $N$ | $\tilde{k}$= 5e3 | $\tilde{k}$= 1e4 |
|---|---|---|
| 20 | 1.61$e$-12 | 1.09$e$-13 |
| 50 | 5.28$e$-11 | 4.39$e$-12 |
| 100 | 2.54$e$-10 | 2.14$e$-11 |

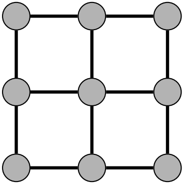
# Asynchronous algorithm: Types of network



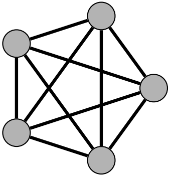
(a) Cycle



(b) Wheel



(c) Grid



(d) Complete

# Asynchronous algorithm: Mean error

Table: Diminishing stepsize: $\tilde{k} = 5e3$

| N | Cycle | Wheel | Grid | Complete |
|---|---|---|---|---|
| 20 | 5.97$e$-1 | 2.45$e$-4 | 9.74$e$-3 | 7.41$e$-5 |
| 50 | 21.17 | 1.67 | 4.78 | 8.99 |
| 100 | 107.76 | 163.71 | 84.9 | 16.86 |

# Asynchronous algorithm: Mean error

Table: Diminishing stepsize: $\tilde{k} = 5e3$

| N | Cycle | Wheel | Grid | Complete |
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| 20 | 5.97$e$-1 | 2.45$e$-4 | 9.74$e$-3 | 7.41$e$-5 |
| 50 | 21.17 | 1.67 | 4.78 | 8.99 |
| 100 | 107.76 | 163.71 | 84.9 | 16.86 |

Table: Diminishing stepsize: $\tilde{k} = 1e4$

| N | Cycle | Wheel | Grid | Complete |
|---|---|---|---|---|
| 20 | 7.69$e$-2 | 7.16$e$-6 | 2.85$e$-4 | 4.82$e$-7 |
| 50 | 11.0 | 1.05$e$-1 | 4.78 | 8.98$e$-3 |
| 100 | 54.2 | 13.31 | 34.5 | 1.72 |

# Asynchronous algorithm: Mean error

Complete $\succ$ Wheel $\succ$ Grid $\succ$ Cycle

Table: Diminishing stepsize: $\tilde{k} = 5e3$

| N | Cycle | Wheel | Grid | Complete |
|---|---|---|---|---|
| 20 | 5.97$e$-1 | 2.45$e$-4 | 9.74$e$-3 | 7.41$e$-5 |
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| 100 | 54.2 | 13.31 | 34.5 | 1.72 |

# Asynchronous algorithm: Confidence interval width

Complete $\succ$ Wheel $\succ$ Grid $\succ$ Cycle

Table: Diminishing stepsize: $\tilde{k} = 5e3$

| N | Cycle | Wheel | Grid | Complete |
|---|---|---|---|---|
| 20 | 2.42$e$-4 | 2.10$e$-7 | 9.51$e$-6 | 2.29$e$-8 |
| 50 | 6.34$e$-3 | 9.25$e$-4 | 2.77$e$-3 | 1.12$e$-4 |
| 100 | 5.88$e$-2 | 8.38$e$-1 | 7.46$e$-2 | 1.65$e$-2 |

Table: Diminishing stepsize: $\tilde{k} = 1e4$

| N | Cycle | Wheel | Grid | Complete |
|---|---|---|---|---|
| 20 | 1.46$e$-5 | 2.54$e$-8 | 1.21$e$-7 | 7.64$e$-10 |
| 50 | 1.13$e$-3 | 5.55$e$-5 | 8.21$e$-4 | 3.67$e$-6 |
| 100 | 9.90$e$-3 | 4.09$e$-3 | 4.94$e$-3 | 1.78$e$-3 |

# Asynchronous algorithm: Confidence interval width

Complete $\succ$ Wheel $\succ$ Grid $\succ$ Cycle

- $p_{\min}/p_{\max}$ : probability of agents performing update
- $\lambda$ : second largest eigenvalue of the expected weight matrix

# Asynchronous algorithm: Confidence interval width

Complete $\succ$ Wheel $\succ$ Grid $\succ$ Cycle

- $p_{\min}/p_{\max}$ : probability of agents performing update
- $\lambda$ : second largest eigenvalue of the expected weight matrix

Table: Number of iteration for concurrence of player's aggregate within an error of 1e-2

| Network | $p_{\min}/p_{\max}$ | $\lambda$ | Iterations |
|---|---|---|---|
| Cycle | 1 | 0.9994 | 177 |
| Grid | 5/7 | 0.3151 | 66 |
| Wheel | 1/19 | 0.1622 | 31 |
| Complete | 1 | 1.0888e-08 | 17 |

# Insights from numerics

- Synchronous algorithm outperforms asynchronous algorithm
- Connectivity graph plays an important role
- Asynchronous algorithm is slow
- Complete $\succ$ Wheel $\succ$ Grid $\succ$ Cycle
  - Depends on the probabilities of agents performing update
  - And the second largest eigenvalue of the expected weight matrix

# References I

# Further extensions

- More general case can be solved

$$\begin{aligned} \text{minimize} \quad & f_i(x_i, \bar{x})), \quad \bar{x} = \sum_{j=1}^{N} x_j \\ \text{subject to} \quad & x_i \in K_i \subseteq \mathbb{R}^n, \\ & g_1(x) \le 0, \ldots, g_m(x) \le 0 \qquad \text{(AggGame)} \end{aligned}$$

  under suitable assumptions on $g_i$.
- Push-sum can be used for mixing (instead of convex combinations)