Numerical schemes for partial differential equations related to optimal stopping time problems

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Joint work with
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Outline

- **I. Diffusion equations with an obstacle term**
- **II. Schemes ... and difficulties**
- **III. A second order scheme**
- **VI. Numerical examples**

I. Motivation

Diffusion equation with an obstacle term:

$$(1)\quad \begin{cases} \min\left(u_t - \frac{\sigma(x)^2}{2}u_{xx} - b(x)u_x + ru,\ u - \varphi(x)\right) = 0, \\ \qquad\qquad\qquad\qquad t > 0,\ x \in \mathbb{R} \\ u(0,x) = \varphi(x), \quad x \in \mathbb{R} \end{cases}$$

Link with stochastic optimal stopping time problem:

$$u(t,x) = \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E}\left[e^{-r\tau}\varphi(X_\tau^{0,x})\right], \quad t \geq 0, x \in \mathbb{R}$$

where

- $\mathcal{T}_{[0,t]}$ is the set of "stopping times" with values in $[0,t]$
- $X_\theta = X_\theta^{0,x}$ satisfies the following SDE :

$$\begin{cases} dX_\theta = b(X_\theta)d\theta + \sigma(X_\theta)dW_\theta, \quad \theta \geq 0 \\ X_0 = x. \end{cases}$$

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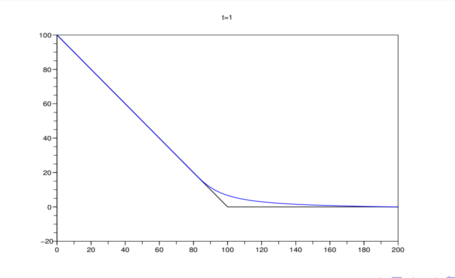
Financial problem: American put option

SDE $X_0 = x \geq 0$ and $\frac{dX_\theta}{X_\theta} = rd\theta + \sigma(X_\theta)dW_\theta$, $(\theta \geq 0)$

COST $\varphi(x) := \max(K - x, 0)$.

$$\Rightarrow \begin{cases} \min(u_t - \frac{x^2\sigma^2(x)}{2}u_{xx} - rxu_x + ru, u - \varphi(x)) = 0, & t \in [0, T],\ x \geq 0 \\ u(0,x) = \varphi(x), & x \geq 0 \end{cases}$$

Figure: american option



Dewynne, Howison, Rupf and Wilmott (1993): "Some mathematical results in the pricing of American options";

- There exists a singular point $x_s(t)$ such that

$$u(t,x) = \varphi(x), \quad x \leq x_s(t), \tag{1}$$

$$u_t(t,x) + \mathcal{A}u(t,x) = 0, \quad x > x_s(t), \tag{2}$$

$$u(t,x) = \varphi(t,x) \text{ and } u_x(t,x) = \varphi_x(t,x), \ \text{ for } x = x_s(t). \tag{3}$$

- $x_s(t) \simeq K - c_0 \log(t)\sqrt{t}$ for small t.
- regularity : u_t, u_{xx} bounded, regular for $x \neq x_s(t)$

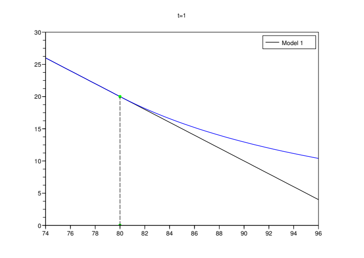


Figure: Zoom around the singular point x_s

Numerical difficulties

- Non linear PDE
- Singular: u_t, u_{xx} bounded but no more at $x = x_s(t)$ (for $t > 0$)
- Singular initial data at $t = 0$: $\max(K - x, 0)$
- Fast variation ($\sqrt{t}$) of the singularity $x_s(t)$ for $t \simeq 0$

A recent work of Reisinger and Forsyth shows that by using the change of variable of the form $v(t, x) = u(t^2, x)$ allows to smoothen the problem at $t = 0$ and partly remove the numerical problems coming from the initial singularity.

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An equivalent PDE (Thanks to R. Eymard)

Using the fact that $u_t \geq 0$, we can show that PDE (1) is equivalent to an other PDE:

$$(2) \begin{cases} \min\left(u_t - \frac{\sigma(x)^2}{2} u_{xx} - b(x)u_x + ru,\ u_t\right) = 0, & t > 0,\ x \in \mathbb{R} \\ u(0,x) = \varphi(x), & x \in \mathbb{R} \end{cases}$$

that is, the obstacle term is replaced by u_t.

It is important that the coefficients r, b, σ does not depend of the time for this equivalence to hold (in the financial context).

Equivalently,

$$(2) \Leftrightarrow \begin{cases} u_t + \min(-\frac{\sigma(x)^2}{2} u_{xx} - b(x)u_x + ru,\ 0) = 0, & t > 0,\ x \in \mathbb{R} \\ u(0,x) = g(x), & x \in \mathbb{R} \end{cases}$$

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All truth passes through three stages :
First, it is ridiculed
Second, it is violently opposed
Third, it is accepted as being self-evident
(Schopenhauer 1788-1860)

Idea of the proof of (1) $\Leftrightarrow$ (2)

1)

$$u_t + \mathcal{A}u \geq 0.$$

$\Big($ Using the semi-martingale property (or semi-DPP) :

$u(t+h,x) \geq \mathbb{E}(e^{-rh}u(t, X_h^{0,x}))$ $\Big)$

2) Notice that $u_t \geq 0$. Indeed,

$$\begin{aligned} u(t+h,x) &= \sup_{\tau \in \mathcal{T}_{[0,t+h]}} \mathbb{E}[g(X_\tau^{0,x})] \\ &\geq \sup_{t \in \mathcal{T}_{[0,t]}} \mathbb{E}[g(X_\tau^{0,x})] = u(t,x). \end{aligned}$$

Hence

$$\min(u_t + \mathcal{A}u,\ u_t) \geq 0.$$

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Idea of the proof of (1) $\Leftrightarrow$ (2) (continued)

3) Suppose that $u_t(t,x) > 0$ (with $t > 0$). Then for $h > 0$ small enough,

$$u(t,x) > u(t-h,x) \geq \varphi(x).$$

Hence one can show that the optimal stopping time $\tau^*_{t,x} := \max\{0 \leq \theta \leq t,\ u(\theta, X^{0,x}_\theta) = \varphi(X^{0,x}_\theta)\}$ satisfies $\tau^*_{t,x} < t$ a.s., from which we can conclude using a DPP and Ito's formula that

$$u_t(t,x) + (\mathcal{A}u)(t,x) = 0.$$

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Application of PDE (2) :

$$\begin{cases} u_t + \min(\mathcal{A}u, 0) = 0, & t > 0,\ x \in \mathbb{R} \\ u(0,x) = \varphi(x), & x \in \mathbb{R} \end{cases}$$

• Related stochastic optimal control problem: u is also obtained as

$$u(t,x) = \sup_{\alpha(.)\in[0,1]} \mathbb{E}\left[e^{-\int_0^t \alpha(s) r\, ds} \varphi(X_t^{0,x,\alpha}) \right]$$

where $X_\theta = X_\theta^{0,x,\alpha}$ is the solution of $X_0 = x$ and

$$dX_\theta = \alpha(\theta)(b(X_\theta)d\theta + \sigma(X_\theta)dW_\theta).$$

• In particular the following DPP holds:

$$u(t_{n+1},x) = \sup_{\alpha(.)\in[0,1]} \mathbb{E}\left[e^{-\int_0^h \alpha(s) r\, ds} u(t_n, X_h^{0,x,\alpha}) \right]$$

• Taking $\alpha \equiv 0$, or $\alpha \equiv 1$ leads to the approximation

$$u(t_{n+1},x) \simeq \max\left(u(t_n,x),\ \mathbb{E}\left[e^{-rh} u(t_n, X_h^{0,x}) \right] \right).$$

II. Schemes and difficulties

- finite difference schemes
- efficient tools for solving non-linear implicit schemes.
- the "1st order barrier"

Simplified obstacle problem: $(\sigma^2/2 = 1,\ b = 0,\ r = 0)$

$$\min(v_t - v_{xx},\ v - \varphi(x)) = 0, \quad t \in (0,T),\ x \in \Omega = (X_{min}, X_{max}),$$
$$v(0,x) = \varphi(x)$$

with dirichlet boundary conditions on $\partial\Omega$.

Explicit finite difference scheme: Mesh $(x_i = ih)$, $(t_n = n\tau)$,

$$\min\left(\frac{u_i^{n+1} - u_i^n}{\tau} - \left(\frac{u_{i-1}^n - 2u_i^n + u_{i-1}^n}{h^2}\right),\ u_i^{n+1} - \varphi(x_i)\right) = 0, \quad 1 \le i \le J$$

with $u_0^{n+1} = u_{J+1}^{n+1} = 0$

... is really explicit ! :

$$\Rightarrow \quad u_i^{n+1} = \max\left(u_i^n - \tau\left(\frac{u_{i-1}^n - 2u_i^n + u_{i-1}^n}{h^2}\right),\ \varphi(x_i)\right)$$

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- **Linear case**

$$v_t - v_{xx} = 0$$

- **Explicit scheme:**

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hence with $k := \dfrac{\tau}{h^2}$:

$$u_i^{n+1} = ku_{i-1}^n + (1 - 2k)u_i^n + ku_{i+1}^n \;\equiv (Su^n)_i$$

- CONSISTENCY: $\dfrac{v^{n+1} - Sv^n}{\tau} = (v_t - v_{xx})(t_n, x_i) + O(\tau) + O(h^2)$

- STABILITY : **CFL condition** $\boxed{\frac{2\tau}{h^2} \le 1} \Rightarrow \|U^{n+1}\|_\infty \le \|U^n\|_\infty$

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- **IMPLICIT scheme:** (for linear case $v_t - v_{xx} = 0$)

$$\frac{u_i^{n+1} - u_i^n}{\tau} - \left(\frac{-u_{i-1}^{\mathbf{n+1}} + 2u_i^{\mathbf{n+1}} - u_{i-1}^{\mathbf{n+1}}}{h^2} \right) = 0 \quad 1 \leq i \leq J$$

$\Rightarrow AU^{n+1} = U^n$, with

$$U^n = \begin{pmatrix} u_1^n \\ \vdots \\ u_J^n \end{pmatrix} \quad A = \begin{bmatrix} 1+2k & -k & & \\ -k & \ddots & \ddots & \\ & & \ddots & -k \\ & & -k & 1+2k \end{bmatrix} \quad \text{and } k := \frac{\tau}{h^2} \geq 0.$$

- CONSISTENCY: idem, $O(\tau) + O(h^2)$
- STABILITY : **NO CFL condition !**

$$A \text{ "}\delta\text{-diag. dominant"} \Rightarrow \|A^{-1}\|_\infty \leq \tfrac{1}{\delta} \leq 1 \Rightarrow \|U^{n+1}\|_\infty \leq \|U^n\|_\infty \quad \forall \tau > 0$$

• **implicit for the nonlinear case:** **Can we do the same ?**

Implicit finite difference scheme

$$\min\left(\frac{u_i^{n+1}-u_i^n}{\tau}-\left(\frac{-u_{i-1}^{n+1}+2u_i^{n+1}-u_{i-1}^{n+1}}{h^2}\right),\ u_i^{n+1}-\varphi(x_i)\right)=0,$$

$$1\le i\le J$$

After multiplication of the left part of the min by $\tau>0$, we get:

$$\min\Big(\underbrace{(1+2k)u_i^{n+1}-ku_{i-1}^{n+1}-ku_{i+1}^{n+1}}_{=(Au^{n+1})_i}-\underbrace{u_i^n}_{\equiv b_i},\ u_i^{n+1}-\underbrace{\varphi(x_i)}_{\equiv g_i}\Big)=0$$

$$\Leftrightarrow \quad \text{find } x=U^{n+1},\quad \min((Ax-b)_i, x_i-g_i)=0,\quad 1\le i\le J$$

• EXISTENCE, STABILITY :

- If $A_{ii}\ge 0$ & A: $\delta\ge 1$-diag. domi. $\Rightarrow \exists! x$
- Moreover $\|U^{n+1}\|_\infty \le \max(\|U^n\|_\infty, \|g\|_\infty)$ $\quad\Rightarrow$ **NO CFL condition.**

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efficient tools for solving non-linear implicit schemes

PB: solve $F(x) := \min(Ax - b, x - g) = 0$.

- Known also as the "linear complementary problem" (when $g = 0$).
- Exist. and Uniq. result based on
  - positivity of all principal minors of A (A is a "P-matrix"[1])
  - the eigenvalues of A (Iain Smears, Lecture Notes, $\sim$2013)
- From the numerical and financial math. point of view, the following approaches have been considered (non exhaustive list)

(1) PSOR ($A = L + U$)

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(4) Splitting (not solving $F = 0$ but rather $x = max(A^{-1}b, g)$).

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Approach (2): Brennan and Schwartz : based on $A = UL$ decomposition with

$$U = tridiag(0, 1, u_i), \quad \text{and} \quad L = tridiag(l_i, d_i, 0), \ (d_i > 0)$$

Formally assumes that

$$\min(ULx - b, x - g) = 0 \quad \overset{(*)}{\Longleftrightarrow} \quad \min(Lx - U^{-1}b, x - g) = 0.$$

Then uses the descent algorithm (cost $O(J)$)

Proposition

(i) The equivalence $(*)$ is true for "profiled" solutions $x \in \mathbb{R}^J$ such that

$$\left\{ \exists i_0, \ x_i = g_i \text{ for } i \leq i_0, \text{ and } x_i > g_i \text{ otherwise} \right\}$$

(ii) Otherwise false (or should be adapted...)

PB: solve $F(x) := \min(Ax - b, x - g) = 0.$

Approach (3): semi-smooth Newton's method

- **Definition.** For $F(x)_i = \min((Ax - b)_i, x_i - g_i)$, let

$$F'(x)_{ij} = \begin{cases} A_{ij} \text{ if } (Ax - b)_i \leq (x - g)_i \\ I_{ij} = \delta_{ij} \text{ otherwise.} \end{cases}$$

- **Newton's algorithm:**

$$x^{k+1} = x^k - F(x^k)^{-1} F(x^k).$$

- More generally, consider for instance Merton's portfolio problem:

$$v(T-t,x) := \operatorname*{ess\,sup}_{\alpha:(t,T)\to\mathcal{K}} \mathbb{E}[\varphi(X_T^{t,x,\alpha})|\mathcal{F}_t], \qquad \boxed{\mathcal{K} \text{ compact}}$$

with $dX_\theta/X_\theta = b(X_\theta,\alpha_\theta)d\theta + \sigma(X_\theta,\alpha_\theta)dW_\theta$, leads to

$$\min_{a\in K}\left(v_t - \frac{1}{2}\sigma^2(x,a)x^2\ v_{xx} - b(x,a)x\ v_x\right) = 0.$$

Implicit finite difference scheme : leads to a matrix A_a and vector b_a depending of the parameter a, and the implicit scheme to solve:

$$\boxed{\min_{a\in K}(A_a x - b_a) = 0, \quad x \in \mathbb{R}^J}$$

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- **Definition 1.** B is a monotone matrix if, equivalently

  (i) $B^{-1} \geq 0$ *componentwise*
  (ii) $BX \geq 0 \Rightarrow X \geq 0$ *for all* X.

- **Definition 2.** B is an M matrix if

  B is diagonal dominant,
  $B_{ii} \geq 0$
  $B_{ij} \leq 0.$

- **Definition 3.** (Mixed matrices) For $a = (a_1, \ldots, a_J) \in \{0,1\}^J$, let

$$\begin{cases} B^a_{ij} := A_{ij},\ b^a_i := b_i \text{ if } a_i = 0 \\ B^a_{ij} := I_{ij},\ b^a_i := g_i \text{ if } a_i = 1 \end{cases}$$

Then $F(x) = \min(Ax - b, x - g) = \min_{a \in \{0,1\}^J}(B^a x - b^a)$

- **Monotony assumption (Mono):**

$$\forall a \in \{0,1\}^J,\ B^a \text{ is a monotone matrix}$$

- **Example:** A is an M-matrix $\Rightarrow$ **(Mono)**

Theorem ((Rust & Santos 04') (O.B., Maroso, Zidani 09'))

Assume **(Mono)**,
(i) *there exists a unique* $x \in \mathbb{R}^J$ *s.t.* $F(x) = 0$;
(ii) $\forall x^0,\ \lim_{k\to\infty} x^k = x$. *(Furthermore* $x^k \leq x^{k+1}$*)*
(iii) *The convergence is in at most* 2^J *iterations.*

Proof: equivalence of Howard's algorithm (1958') and Newton's method.

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Obstacle pb: the convergence is in at most J *iterations !*

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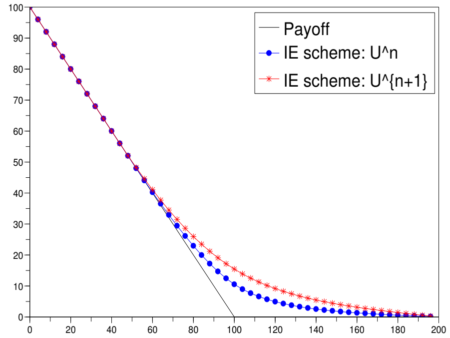
Obstacle pb: the convergence is in at most J iterations !

Application to american options: (OB, Maroso, Zidani 2009)
Limitation of the total number's of Newton's iteration := bounded by the number of mesh points where the value "takes off" the payoff function.

Figure: Two successive iteration for the american option



The first order "barrier"

- Can we improve the error in time from $O(\tau)$ to $O(\tau^2)$ or better ?
- **(Godunov's Th.)** For $u_t + bu_x = 0$, monotone linear schemes $u_i^{n+1} = \sum a_j u_{i+j}^n$ ($a_j \geq 0$) are limited to at most first order.
- For other reasons (see later on..) for non linear diffusion problems some limitations hold, and we are led to try implicit schemes.

- Let us try a Crank - Nicolson scheme (RK2):

$$S_j^{1,n}(u) := \min\left(\frac{u_i^{n+1} - u_i^n}{\tau} + \frac{1}{2}(Au^n + Au^{n+1})_i,\ u_i^{n+1} - g_i\right) = 0,$$

$$1 \leq i \leq J$$

For A: M-matrix, we can solve it by a Newton's Algorithm.

- Numerical experiments show
  - - using $\tau = h$, stable, **second order** (**ObservationA**)
  - - using $\tau = 10h$, stable, but only **first order** (**ObservationB**)

What is wrong ??

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- - using $\tau = h$, stable, **second order** (**ObservationA**)
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What is wrong ??

Consistency analysis:

Notice that for a regular v and $v_i^n = v(t_n, x_i)$:

$$\frac{v_i^{n+1} - v_i^n}{\tau} + \frac{1}{2}(Av^n + Av^{n+1})_i$$
$$= v_t(t_{n+1/2}, x_i) + Av(t_{n+1/2}, x_i) + O(\tau^2)$$
$$= (v_t - v_{xx})(t_{n+1/2}, x_i) + O(\tau^2 + h^2)$$

Hence

$$S_j^{1,n}(v) = \min((v_t - v_{xx})_i^{n+1/2} + O(\tau^2 + h^2),\ v_i^{n+1} - g_i)$$
$$= \min((v_t - v_{xx})_i^{n+1/2},\ v_i^{n+1} - g_i) + O(\tau^2 + h^2)$$

But we cannot assume that $v_t - v_{xx} \equiv 0$!

In general

$$S_j^{1,n}(v) = \min((v_t - v_{xx})_i^{n+1/2},\ v_i^{n+1} - g_i) + O(\tau^2 + h^2)$$
$$= \min((v_t - v_{xx})_i^{n+1} - \frac{\tau}{2}\partial_t(v_t - v_{xx}),,\ v_i^{n+1} - g_i) + O(\tau^2 + h^2)$$

so we have in general **only first order consistency**, in the sense

$$S_j^{1,n}(v) = \min(v_t - v_{xx}, v - \varphi(x_j))_{(t_{n+1},x_i)} + O(\tau + h^2)$$

More generally, consider a high order scheme for the diffusion part, i.e. such that, for instance,

$$\frac{v^{n+1} - S(u^n)}{\tau} = (v_t - v_{xx})(t_n, x_i) + O(\tau^q) + O(h^2)$$

for some $q \geq 2$. (Ex: Weak Taylor schemes, Platen's scheme, Kloeden and Platen 1995,...)
The corresponding scheme for the obstacle equation,

$$u_i^{n+1} = \max(S(u^n)_i, g_i)$$

and equivalent to

$$\min\left(\frac{v^{n+1} - S(u^n)}{\tau},\ u_i^{n+1} - g_i\right) = 0$$

will **not be high order** in time for the obstacle PDE.

Then how the CN scheme could be of second order ?!!

Looking at PDE (2), an other scheme can be (Scheme (2) for PDE (2))

$$\mathcal{S}_j^{2,n} := \min\left(\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{2}(Au^{n+1} + Au^n)_j,\ \frac{u_j^{n+1} - u_j^n}{\tau} \right) = 0 \qquad 1 \le j \le J \tag{4}$$

This is second order in time consistent, at time $t = t_{n+1/2}$!

Numerical experiments show:

- - using $\tau = h$, stable, **second order**
- - using $\tau = 10h$, switch to **first order** (**ObservationC**)

Some explanations...

Related to observations (**A**) and (**B**):

Lemma

(i) If $u^{n+1} \geq u^n$ for the scheme (1) (CN-obstacle scheme for PDE (1), with the constraint $u_j^n \geq g_j$), then u^{n+1} is also solution of the scheme (2) : hence both schemes give identical values.

(ii) This is the case in particular when
- the matrix $I - \frac{\tau}{2}A$ is positive componentwise (which is the case under an appropriate CFL condition of the form $\frac{\tau}{h^2} \leq c_0$)
- the matrix $I + \frac{\tau}{2}A$ is an M-matrix (always the case here).

Related to observation (**C**): the lost of the second order behavior can be explained by the fact that the CN scheme does not have very nice stability behavior for bad CFL numbers.

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Related to observation (**C**): the lost of the second order behavior can be explained by the fact that the CN scheme does not have very nice stability behavior for bad CFL numbers.

III. A BDF Scheme

In order to get the right consistency error, we use the following 3-point formula:

$$\frac{3v_i^{n+1} - 4v_i^n + v_i^{n-1}}{2\tau} \simeq v_t(t_{n+1}, x_i) + O(\tau^2)$$

Known also as a "Backward Difference Formula" of second order

Hence the corresponding implicit BDF2 scheme, for $n \geq 1$:

$$\mathcal{H}_j^{n+1}(u) :\equiv \min\left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\tau} + (Au^{n+1})_j,\ u_j^{n+1} - g_j\right) = 0$$

$$1 \leq j \leq J$$

- Multi Step scheme, needs u^0 and u^1 estimate to start
- For the linear part, known also as the "Gear" Scheme
- Scheme already proposed by Oosterlee (2003) together with multigrid idea for 2d problems.

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Proposition

(i) Scheme is second-order consistent in time and space.
(ii) Scheme is fastly implementable for $B = A + \frac{2\tau}{3}$: M-matrix.

Proof of (ii): Equivalent scheme:

$$\min\left((I + \frac{2\tau}{3}A)u^{n+1} - \frac{4}{3}u^n + \frac{1}{3}u^{n-1},\ u^{n+1} - g\right) = 0$$

$$1 \leq j \leq J$$

• Other temptative consistent schemes can be proposed, but the miracle here is that BDF2 for the obstacle problem can be shown to be stable - and convergent ! (although non monotone)

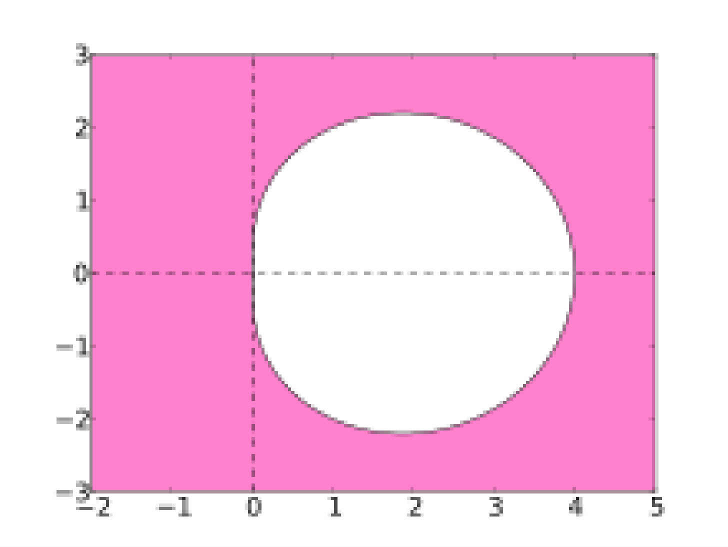
***A*-stability :** for the ODE

$$\dot{y} = \lambda y$$

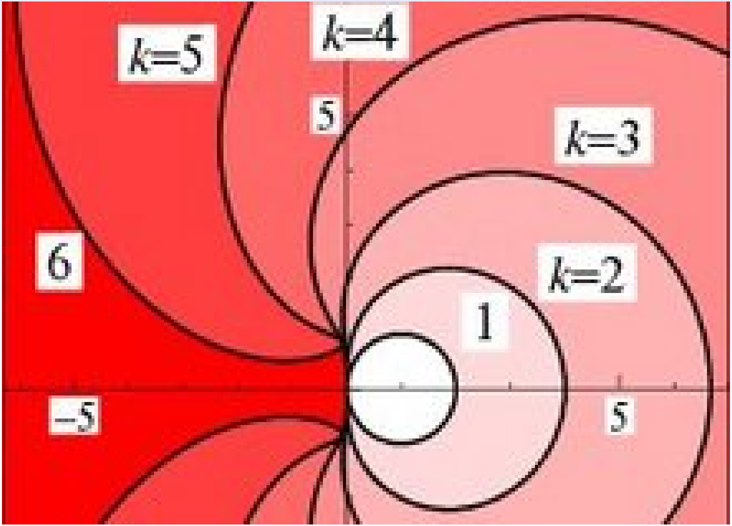
we look for which $\lambda \in \mathbb{C}$ the scheme is stable. If it is stable for all $Re(\lambda) \leq 0$ then we say the scheme is A-stable.

- BDF2 is A-stable
- BDFk , $k \geq 3$ are not A-stable.

Wikipedia - Stability region for BDF2:



Ernst Hairer and Gerhard Wanner (2010), Scholarpedia, 5(4):4591.



Numerical experiments show:

- using $\tau = h$: stable, second order
- using $\tau = 10h$: (incondtionnally) stable, second order

$\Rightarrow$ very efficient !

STABILITY AND CONVERGENCE RESULT

For a given vector $x = (x_j)_{1 \le j \le J}$, let

$$\|x\|_2 = (\sum_{j=1}^{J} |x_j|^2)^{1/2} \tag{5}$$

$$N_h(x) := \left(\sum_{j=1}^{J+1} |\frac{x_j - x_{j-1}}{h}|^2 \right)^{1/2} \tag{6}$$

(with the convention $x_0 := 0$ and $x_{J+1} := 0$).

• Assume the following coercivity on A: there exists constants $\eta > 0$, $\gamma \ge 0$ such that:

$$\langle e, Ae \rangle \ge \eta N_h(e)^2 - \gamma \|e\|_2^2. \tag{7}$$

• Coercivity holds for the matrix A coming from the F.D. approximation of $\mathcal{A}v = -a(x)v_{xx} + b(x)v_x + c(x)v$ with $a(x) \ge \eta > 0$ and $a \in Lip$, $b, c \in L^\infty$.

Scheme

$$\mathcal{H}_j^{n+1}(u) :\equiv \min\left(\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\tau} + (Au^{n+1})_j,\ u_j^{n+1} - g_j\right) = 0$$

Perturbed scheme for the exact solution $v_j^n = v(t_n, x_j)$:

$$\mathcal{H}_j^{n+1}(u) :\equiv \min\left(\frac{3v_j^{n+1} - 4v_j^n + v_j^{n-1}}{2\tau} + (Av^{n+1})_j - \epsilon_j^n,\ v_j^{n+1} - g_j\right) = 0$$

Error: $e^n := v^n - u^n$

Proposition (Stability estimate)

Let $\tau > 0$ be sufficiently small. There exists C_1 independent of n, for all $t_n \leq T$

$$\|e^n\|_2^2 + \tau \sum_{k=1}^{n} N_h(e^k)^2 \leq C_1\left(\|e^0\|_2^2 + \|e^1\|_2^2 + \tau \sum_{k=1,\ldots,n} \|\epsilon_n\|_2^2\right).$$

For solution v with bounded v_{xx} and v_t (typical regularity for $t > 0$), with an "isolated" singularity $x = x_s(t)$ we can show that $\epsilon_j^n = O(1)$ in worst case (otherwise high-order), therefore $\|\epsilon_n\|_2^2 = O(1)$ and in the end, $\|e^n\|_2^2 = O(1)$.
As a consequence:

Theorem

For the BDF2 implicit obstacle scheme, it holds

$$\|e^n\|_{L^2} \simeq (h\|e^n\|_2^2)^{1/2} = O(h^{1/2})$$

STABILITY : ELEMENTS OF THE PROOF

Let $\langle x, y\rangle$ denote the usual scalar product on $\mathbb{R}^J$.

Lemma

For any matrix B, the following equivalence holds:

$$\min(Bx-b, x-g)=0 \quad \Leftrightarrow \quad x \geq g \text{ and } \Big(\langle Bx-b, v-x\rangle \geq 0,\ \forall v \geq g\Big)$$

Proof in the case B is a positive definite symmetric matrix:

$$\begin{aligned}\min(Bx-b, x-g)=0 \quad &\Leftrightarrow \quad x \text{ solves } \min_{x\geq g} \frac{1}{2}\langle x, Bx\rangle - \langle b, x\rangle \\ &\Leftrightarrow \quad x \geq g \text{ and } \Big(\langle Bx-b, v-x\rangle \geq 0,\ \forall v \geq g\Big)\end{aligned}$$

Let

$$B := I + \frac{2\tau}{3}A,$$

and vectors

$$b_u := \frac{4}{3}u^n - \frac{1}{3}u^{n-1}$$
$$b_v := \frac{4}{3}v^n - \frac{1}{3}v^{n-1}$$

Equation in u^{n+1} is equivalent to

$$\min(Bu^{n+1} - b_u, u^{n+1} - g) = 0$$

By the Lemma, this is equivalent to $u^{n+1} \geq g$ and

$$\left\langle Bu^{n+1} - b_u,\ w - u^{n+1} \right\rangle \geq 0, \quad \forall w \geq g. \tag{8}$$

Similarly, equation for v^{n+1} is equivalent to $v^{n+1} \geq g$ and

$$\left\langle Bv^{n+1} - (b_v + \tau\epsilon^n),\ w - v^{n+1} \right\rangle \geq 0, \quad \forall w \geq g. \tag{9}$$

Plugging $w = v^{n+1}$ in (8) and $w = u^{n+1}$ into (9) gives

$$\left\langle Be^{n+1} - \frac{4}{3}e^n + \frac{1}{3}e^{n-1} - \frac{2\tau}{3}\epsilon^n,\ e^{n+1}\right\rangle \leq 0$$

and therefore

$$\left\langle 3e^{n+1} - 4e^n + e^{n-1},\ e^{n+1}\right\rangle + 2\tau\langle e^{n+1}, Ae^{n+1}\rangle \leq 2\tau\langle \epsilon^n,\ e^{n+1}\rangle.$$

Using the coercivity of A, the argument is then technical but "classical" as for the analysis of BDF2 scheme for a parabolic linear problem.

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DONE !

IV. Numerical examples

MODEL TEST PROBLEMS:

$$\min\left(v_t - \frac{\lambda^2}{2}x^2 v_{xx} - rxv_x + rv,\ v - \varphi(x)\right) = f(t,x)$$
$$v(0,x) = \varphi(x).$$

- Set $x_s(t) := K - c_0\sqrt{t}$
- We construct explicit $v(t,x)$ on $x \in [0, X_{max}]$ s.t.

  (*i*) $v(t,x) = \varphi(x) = K - x$ for $x \leq x_s(t)$,

  (*ii*) $v(t,x) > \varphi(x) = \max(K-x, 0)$ for $x \in]x_s(t), X_{max}]$,

  (*iii*) v is at least C^1 on $[0, X_{max}]$,

  (*iv*) $v(t, X_{max}) = 0$.
- **MODEL 1:** s.t. u_{xx} bounded, regular away from $x = x_s(t)$.
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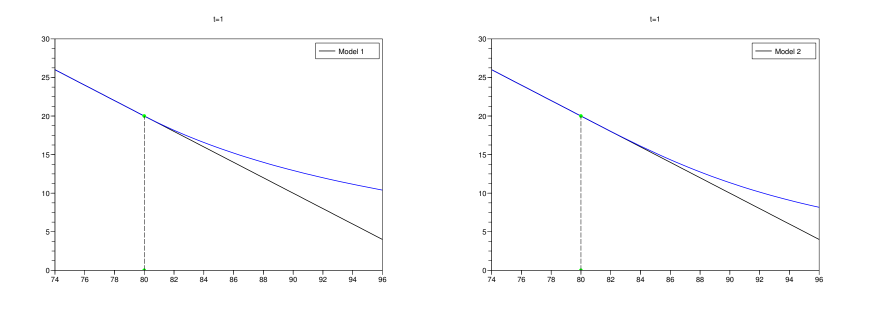

Figure: Zooming around the singular point $(x_s, g(x_s))$ for model 1 (left) and 2 (right).

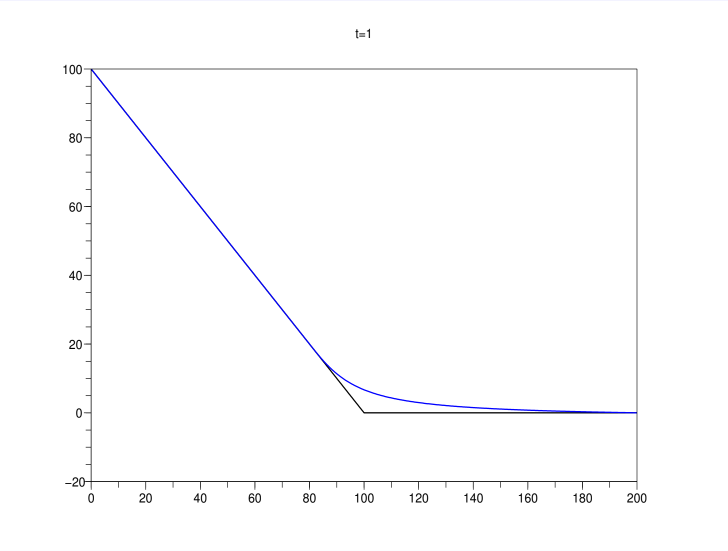

Figure: Model 2.

Numerical examples

Crank Nicolson schemes

Mesh		Error L^1		Error L^2		Error L^∞		CPU time
J	N	error	order	error	order	error	order	
80	80	1.74E-02	-	2.16E-02	-	4.49E-02	-	0.14
160	160	3.15E-03	2.47	3.73E-03	2.54	5.25E-03	3.10	0.25
320	320	8.42E-04	1.91	9.87E-04	1.92	1.39E-03	1.92	0.52
640	640	2.23E-04	1.92	2.58E-04	1.93	3.56E-04	1.97	1.54
1280	1280	6.06E-05	1.88	6.96E-05	1.89	9.15E-05	1.96	6.65
2560	2560	1.66E-05	1.86	1.91E-05	1.86	2.69E-05	1.77	37.66

Table: EDP (1), Crank-Nicolson scheme with $N = J$ ($\tau \equiv h$)

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	1.83E-02	-	2.47E-02	-	4.90E-02	-	0.05
160	16	4.67E-03	1.97	7.23E-03	1.77	2.18E-02	1.17	0.07
320	32	1.68E-03	1.48	2.97E-03	1.28	9.90E-03	1.14	0.13
640	64	6.89E-04	1.28	1.38E-03	1.11	4.75E-03	1.06	0.38
1280	128	2.99E-04	1.20	6.59E-04	1.06	2.31E-03	1.04	1.99
2560	256	1.37E-04	1.13	3.22E-04	1.03	1.14E-03	1.02	16.04

Table: EDP (1), Crank-Nicolson scheme with $N = J/10$ ($\tau \gg h$)

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.68E-02	-	3.11E-02	-	4.90E-02	-	0.25
160	16	6.91E-03	1.96	8.28E-03	1.91	1.14E-02	2.11	0.13
320	32	2.53E-03	1.45	2.97E-03	1.48	3.91E-03	1.54	0.18
640	64	9.35E-04	1.43	1.11E-03	1.43	1.50E-03	1.38	0.41
1280	128	3.74E-04	1.32	4.44E-04	1.32	6.07E-04	1.31	1.66
2560	256	1.56E-04	1.26	1.86E-04	1.26	2.56E-04	1.25	11.36

Table: EDP (2) - Crank-Nicolson scheme, $N = J/10$

BDF schemes (American option problem)

BDF2 implicit obstacle scheme:

$$\min\left(\frac{3u_j^{n+1}-4u_j^n+u_j^{n-1}}{2\tau}+(Au^{n+1}+q(t_{n+1}))_j,\ u_j^{n+1}-g_j\right)=0$$

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	80	1.72E-02	-	2.14E-02	-	4.48E-02	-	0.13
160	160	3.02E-03	2.51	3.57E-03	2.58	5.06E-03	3.15	0.26
320	320	7.93E-04	1.93	9.30E-04	1.94	1.32E-03	1.93	0.49
640	640	2.06E-04	1.94	2.40E-04	1.96	3.39E-04	1.97	1.54
1280	1280	5.46E-05	1.92	6.30E-05	1.93	8.75E-05	1.95	6.43
2560	2560	1.44E-05	1.93	1.67E-05	1.92	2.64E-05	1.73	36.70

Table: BDF2-implicit scheme ($N = J$)

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	8.23E-03	-	1.25E-02	-	3.59E-02	-	0.06
160	16	9.61E-04	3.10	1.28E-03	3.29	2.20E-03	4.02	0.09
320	32	4.18E-04	1.20	5.41E-04	1.24	8.83E-04	1.32	0.13
640	64	1.54E-04	1.44	1.92E-04	1.49	2.99E-04	1.56	0.32
1280	128	4.77E-05	1.69	5.82E-05	1.73	8.71E-05	1.78	1.14
2560	256	1.14E-05	2.07	1.39E-05	2.06	2.01E-05	2.12	6.21

Table: BDF2-implicit scheme, with high CFL numbers.

BDF3 implicit obstacle scheme:

$$\min\left(\frac{\frac{11}{6}u_j^{n+1} - 3u_j^n + \frac{3}{2}u_j^{n-1} - \frac{1}{3}u_j^{n-2}}{\tau} + (Au^{n+1})_j,\right.$$

$$\left. u_j^{n+1} - g_j\right) = 0, \quad \text{for } n \geq 2$$

- No stability estimate !
- test on a more regular data (MODEL 2) to see the order
- changing A matrix to get 4-th order approximation in space
- Use **smoother initial data** $\varphi(x) = v(t_0, x)$, $t_0 > 0$ to avoid initial singularity

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.56E+00	-	5.93E-01	-	1.94E-01	-	0.10
160	16	6.48E-01	1.98	1.48E-01	2.00	5.11E-02	1.93	0.11
320	32	1.77E-01	1.87	4.00E-02	1.89	1.33E-02	1.95	0.24
640	64	4.72E-02	1.90	1.06E-02	1.92	3.41E-03	1.96	0.48
1280	128	1.25E-02	1.92	2.77E-03	1.94	8.60E-04	1.99	1.28
2560	256	3.24E-03	1.95	7.14E-04	1.96	2.16E-04	2.00	4.64
5120	512	8.06E-04	2.01	1.78E-04	2.00	5.39E-05	2.00	19.30

Table: (Model 1) BDF2 implicit scheme with 4th order spatial approximation, using high CFL numbers.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.50E+00	-	5.99E-01	-	2.09E-01	-	0.09
160	16	6.42E-01	1.96	1.48E-01	2.01	5.34E-02	1.96	0.13
320	32	1.69E-01	1.93	3.85E-02	1.95	1.35E-02	1.99	0.24
640	64	4.41E-02	1.94	9.96E-03	1.95	3.36E-03	2.00	0.53
1280	128	1.15E-02	1.94	2.57E-03	1.96	8.43E-04	2.00	1.66
2560	256	2.96E-03	1.96	6.55E-04	1.97	2.11E-04	2.00	7.27
5120	512	7.33E-04	2.01	1.63E-04	2.01	5.28E-05	2.00	39.23

Table: (Model 1) BDF3 - third order - implicit scheme with 4th order spatial approximation, using high CFL numbers.

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.53E-01	-	4.23E-02	-	9.90E-03	-	0.10
160	16	5.02E-02	2.34	8.54E-03	2.31	2.02E-03	2.29	0.15
320	32	1.09E-02	2.20	1.86E-03	2.20	4.37E-04	2.21	0.26
640	64	2.58E-03	2.08	4.39E-04	2.08	1.04E-04	2.07	0.60
1280	128	6.24E-04	2.05	1.06E-04	2.05	2.50E-05	2.05	1.60
2560	256	1.54E-04	2.02	2.62E-05	2.02	6.16E-06	2.02	5.65
5120	512	3.81E-05	2.01	6.48E-06	2.01	1.53E-06	2.01	24.37

Table: (Model 2) **BDF2** implicit scheme with 4th order spatial approximation, using high CFL numbers, and $t_0 = 0.3$. (SMOOTHER INITIAL DATA)

Mesh		Error L^1		Error L^2		Error L^∞		time(s)
J	N	error	order	error	order	error	order	
80	8	2.51E-02	-	4.42E-03	-	1.21E-03	-	0.11
160	16	3.65E-03	2.78	6.48E-04	2.77	1.76E-04	2.78	0.14
320	32	6.00E-04	2.61	1.04E-04	2.64	2.62E-05	2.75	0.25
640	64	8.56E-05	2.81	1.46E-05	2.83	3.55E-06	2.89	0.65
1280	128	1.09E-05	2.97	1.83E-06	3.00	4.40E-07	3.01	2.11
2560	256	1.42E-06	2.94	2.32E-07	2.98	5.52E-08	2.99	8.96
5120	512	1.92E-07	2.89	3.02E-08	2.94	7.00E-09	2.98	47.05

Table: (Model 2) **BDF3** implicit scheme with 4th order spatial approximation, using high CFL numbers, and $t_0 = 0.3$. (SMOOTHER INITIAL DATA)

CONCLUSION

Errors in terms of number of operations N_{opt}:

BDF2 using $N \equiv J$:

- $N_{opt} \equiv N \cdot J \equiv N^2$,
- observed error : $err = O(\tau^2) = O(\frac{1}{N^2}) = O(\frac{1}{N_{opt}})$.
- theoretical bound : $err_{L^2} = O(h^{1/2}) = O(\frac{1}{N_{opt}^{1/4}})$.

Tree method with N levels (Lamberton 2015), constant coefficient case:

- N levels $\Rightarrow N_{opt} \equiv N^2/2$
- observed error : $err = O(\frac{1}{N}) = O(\frac{1}{N_{opt}^{1/2}})$.
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CONCLUSION 2

Related ongoing works:

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- Implicit schemes for nonlinear diffusion problems (with A. Picarelli, C. Reisinger)

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