Duality-based error estimates for some approximation schemes for optimal investment problems

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Introduction

An optimal investment problem is considered

$$v(t,x) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[U(X^{\alpha}_{t,x}(T))\Big] \qquad x \geq 0.$$

By duality arguments (Kramkov-Schachermayer ('99)):

$$v(t,x) = \inf_{y \geq 0} \Big\{\tilde{v}(t,y) + xy\Big\}$$

where $\tilde{v}$ is the value function associated to the "dual problem".

Let V be a numerical approximation of v.

AIM: Exploit the duality relation in order to obtain new estimates for the error $\|v - V\|_{\infty}$.

Outline of the talk

Model and preliminary results

Model and preliminary results

Consider a model similar to Cuoco-Liu ('00):

$$\begin{cases} dX(s) & = X(s)\Big(r_s + \alpha_s(b_s - r_s) + g(s,\alpha_s)\Big)ds + \alpha_s\sigma_s X(s)d\mathcal{B}_s \\ X(t) & = x \end{cases} \tag{1}$$

- Controls: $\mathcal{A} := \{\text{Progr. meas. processes} : \alpha(\cdot) \in A \text{ a.s.}\}$; A compact and convex set;
- $\mathcal{B}_\cdot$ one-dimensional Brownian motion;
- $\sigma \neq 0, r, b$ bounded and $1/2$-Hölder continuous in t;
- g concave and Lip. continuous in α, $1/2$-Hölder cont. in t;

$\rightsquigarrow$ $X^{\alpha}_{t,x}(\cdot)$: unique solution of (1) associated with $\alpha \in \mathcal{A}$.

Model and preliminary results

Let $T > 0$ and $U \in C^1([0, +\infty); \mathbb{R})$ a utility function:

- U is concave, strictly increasing and bounded.

The value function associated with the optimal investment problem is defined by:

$$v(t,x) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[U(X^{\alpha}_{t,x}(T))\Big] \qquad x \geq 0,\ t \in [0,T].$$

Model and preliminary results

Let us introduce the dual dynamics:

$$\begin{cases} dY(s) &= -\left(r_s + \tilde{g}(s,\nu_s)\right)Y(s)ds + (\sigma_s)^{-1}Y(s)(r_s - b_s - \nu_s)d\mathcal{B}_s \\ Y(t) &= y \end{cases} \qquad (2)$$

where

- $\mathcal{V} := \{\text{Progr. meas. processes} : \nu(\cdot) \in \Gamma \text{ a.s.}\}$;
- $\tilde{g}(t,\nu) := \sup_{a\in A}\{g(t,a) - a\nu\}$;

$\rightsquigarrow$ $Y^{\nu}_{t,y}(\cdot)$: unique solution of (2) associated with $\nu \in \mathcal{V}$.

Model and preliminary results

Let

$$\tilde{U}(y) := \sup_{x \geq 0} \left\{ U(x) - xy \right\}$$

and

$$\tilde{v}(t,y) := \inf_{\nu \in \mathcal{V}} \mathbb{E}\left[\tilde{U}(Y^{\nu}_{t,y}(T)) \right].$$

Theorem (Cuoco-Liu ('00), Rogers ('02))

Let the assumptions considered above be satisfied. Then

$$v(t,x) = \inf_{y \geq 0} \left\{ \tilde{v}(t,y) + xy \right\}$$

for any $t \in [0,T]$ *and* $x \geq 0$*.*

Dynamic programming approach

- v satisfies the Dynamic Programming Principle (DPP):

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[v(t+\theta, X^{\alpha}_{t,x}(t+\theta)) \Big]$$

for any $[0, T-t]$-valued stopping time θ;

- v is the unique continuous viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{cases} -\partial_t v + H(t,x,\partial_x v, \partial_{xx} v) = 0 & \text{in } [0,T) \times \mathbb{R}^+ \\ v(T,x) = U(x) & \text{in } \mathbb{R}^+ \end{cases}$$

where

$$H(t,x,p,Q) := \inf_{a \in A} \Big\{ \big(-r_t + a(r_t - b_t) + g(t,a) \big) xp - \frac{1}{2}(a\sigma_t)^2 x^2 Q \Big\}.$$

Error estimates

Let V be the numerical approximation of v obtained by a scheme

$$\mathcal{S}_{\Delta t,\Delta x}(t_n, x, V(t_n,\cdot), [V]) = 0 \qquad n = N, \dots, 0.$$

RESULTS BASED ON PDE TECHNIQUES:
Consistency error: for any smooth function f one has

$$\mathcal{S}_{\Delta t,\Delta x}(t_n, x, f(t_{n+1},\cdot), [f]) = -\partial_t f + H(t, x, \partial_x f, \partial_{xx} f) + \text{consistency error}$$

$$+$$

"Shaking coefficients": perturbation of the problem for providing a smooth super-sol. of the equation (lower bound) or scheme (upper bound)

(Krylov '00; Barles-Jakobsen '02,'05,'07; Debrabant-Jakobsen '13)

USING DUAL STRUCTURE: "shaking coefficients" + direct estimates. (See also Rogers '13)

A class of approximation schemes

A class of approximation schemes

Given $N \geq 1$, let us introduce a time and control discretization:

$$h = T/N \qquad \text{and} \qquad t_n = nh \quad (n = 0, \dots, N)$$

$$\mathcal{A}_h := \left\{ \alpha \in \mathcal{A} : \alpha(s) \equiv \sum_{i=0}^{N-1} a_i \mathbb{1}_{s \in [t_i, t_{i+1})} \text{ s.t. } a_i \in A \quad i = 0, \dots, N-1 \right\}.$$

Consider:

$$v_h(t,x) := \sup_{\alpha \in \mathcal{A}_h} \mathbb{E}\left[U(X^{\alpha}_{t,x}(T)) \right].$$

A class of approximation schemes

The Dynamic Programming Principle for υ_h reads:

$$\upsilon_h(t_n, x) = \sup_{a \in A} \mathbb{E}\Big[\upsilon_h(t_{n+1}, X^a_{t_n,x}(t_{n+1}))\Big]$$

For any f sufficiently smooth, let us consider an approximation scheme $\mathcal{S}$ such that

$$\mathbb{E}\Big[f(t_{n+1}, X^a_{t_n,x}(t_{n+1}))\Big] \approx \mathcal{S}[f(t_{n+1}, \cdot)](t_n, x, a).$$

The approximated value function V is recursively defined by

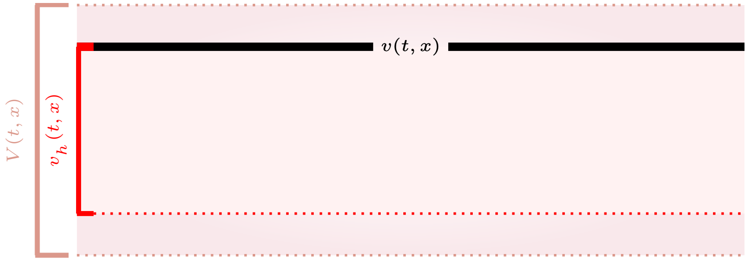
$$\begin{cases} V(t_N, x) & = U(x) \\ V(t_n, x) & = \sup_{a \in A} \mathcal{S}[V(t_{n+1}, \cdot)](t_n, x, a) \qquad n = N-1, \dots, 0. \end{cases}$$

Direct error estimates

Let us assume that, passing through a regularization procedure if necessary, one has

$$
\begin{aligned}
&\sup_{n=0,\dots,N} \left| v_h(t_n,x) - V(t_n,x) \right| \\
&\le \sup_{n=0,\dots,N-1} \sup_{a\in A} \left| \mathbb{E}\left[v_h(t_{n+1}, X^a_{t_n,x}(t_{n+1})) \right] - \mathcal{S}[V(t_{n+1},\cdot)](t_n,x,a) \right| \\
&\le \ldots \le Ch^p \qquad\qquad p > 1/6.
\end{aligned}
$$

Moreover (Krylov '99): $0 \le v(t_n,x) - v_h(t_n,x) \le Ch^{1/6}$.



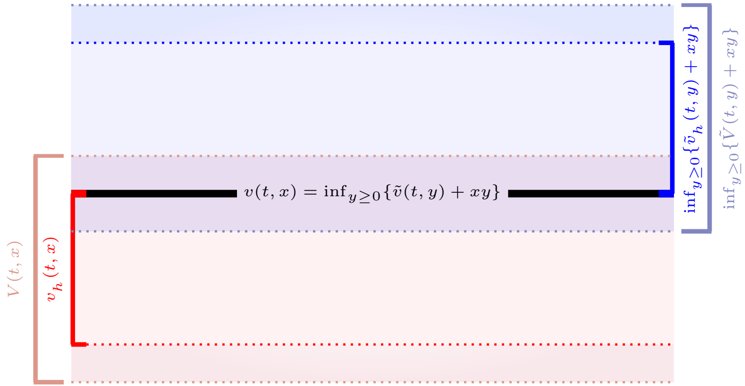
Duality based error estimates

Defining the approximated dual value function $\tilde{V}$ one has

$$\sup_{n=0,\dots,N} \left| \tilde{v}_h(t_n, x) - \tilde{V}(t_n, x) \right| \leq Ch^p,$$

and

$$-Ch^{1/6} \leq \tilde{v}(t_n, x) - \tilde{v}_h(t_n, x) \leq 0.$$



Duality based error estimates

The previous arguments provide:

- a posteriori error estimates based on the quantity

$$\left| V(t,x) - \inf_{y \geq 0} \left\{ \tilde{V}(t,y) + xy \right\} \right|.$$

- A priori error estimates IF some duality relation holds for the approximated value functions V and $\tilde{V}$.

A Markov chain approximation

A Markov chain approximation

Given an integer $N \geq 1$ let

$$h = T/N \qquad \text{and} \qquad t_n = nh \quad (n = 0, \ldots, N);$$

$$\mathcal{A}_h \subseteq \mathcal{A} \qquad \text{piecewise constant controls.}$$

Taken $\alpha \in \mathcal{A}_h$, consider the Euler-Maruyama approximation:

$$X^{\alpha,EM}(t_{i+1}) = X^{\alpha,EM}(t_i)\Big(1 + r_i h + \alpha_i(b_i - r_i)h + g(t_i,\alpha)h + \alpha_i\sigma_i\Delta\mathcal{B}_i\Big)$$

where $\Delta\mathcal{B}_i$ i.i.d. such that $\Delta\mathcal{B}_i \sim \sqrt{h}\mathcal{N}(0,1)$ for $i = 0, \ldots, N-1$.

$$V_{EM}(t_n, x) := \sup_{\alpha \in \mathcal{A}_h} \mathbb{E}\Big[U(X^{\alpha,EM}_{t_n,x}(T))\Big].$$

A Markov chain approximation

V_{EM} satisfies the following DPP:

$$
\begin{aligned}
V(t_n, x) &= \sup_{a\in A} \mathbb{E}\Big[V(t_{n+1}, X^{a,EM}_{t_n,x}(t_{n+1}))\Big] \\
&= \sup_{a\in A} \int_{-\infty}^{+\infty} V\Big(t_{n+1}, x(1 + h(r_n + a(b_n - r_n) + g(t_n, a)) + a\sigma_n\sqrt{h}y)\Big)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy \\
&\approx \sup_{a\in A} \sum_{i=1}^{M} \lambda_i V\Big(t_{n+1}, x(1 + h(r_n + a(b_n - r_n) + g(t_n, a)) + a\sigma_n\sqrt{h}\xi_i)\Big) \quad \left(\begin{array}{c}\text{Gauss-Hermite}\\ \text{quadrature formula}\end{array}\right) \\
&= \sup_{a\in A} \underbrace{\mathbb{E}\Big[V\Big(t_{n+1}, \underbrace{x(1 + h(r_n + a(b_n - r_n) + g(t_n, a)) + a\sigma_n\sqrt{h}\zeta_n}_{X^{a,h}_{t_n,x}(t_{n+1})}\Big)\Big]}_{=:\mathcal{S}[V(t_{n+1},\cdot)](t_n,x,a)} \quad \left(\lambda_i \geq 0, \sum_{i=1}^{M}\lambda_i = 1\right)
\end{aligned}
$$

where $\{\zeta_n\}_{n=0,\dots,N-1}$ i.i.d. random variables such that for any $n = 0, \dots, N-1$

$$\mathbb{P}(\zeta_n = \xi_i) = \lambda_i, \quad i = 1, \dots, M.$$

A Markov chain approximation

The scheme we consider is recursively defined by

$$\begin{cases} V(t_N, x) & = U(x) \\ V(t_n, x) & = \sup_{a \in A} \mathbb{E}\Big[V\Big(t_{n+1}, X^{a,h}_{t_n,x}(t_{n+1})\Big)\Big] \qquad n = N-1, \dots, 0 \end{cases}$$

One has

$$V(t_n, x) = \sup_{\alpha \in \mathcal{A}_h} \mathbb{E}\Big[U(X^{\alpha,h}_{t_n,x}(T))\Big]$$

Example $M = 2$.

$$\xi_i = \pm 1 \quad \text{and} \quad \lambda_i = 1/2 \qquad \text{for } i = 1, 2.$$

$\Rightarrow$ Semi-Lagrangian (SL) scheme of Camilli-Falcone ('95).

A Markov chain approximation

The same scheme is used to approximate the dual problem:

$$\begin{cases} \tilde{V}(t_N, y) = & \tilde{U}(y) \\ \tilde{V}(t_n, y) = & \inf_{\nu \in \Gamma} \mathbb{E}\Big[\tilde{V}\Big(t_{n+1}, Y^{\nu,h}_{t_n,y}(t_{n+1})\Big)\Big] \qquad n = N-1, \ldots, 0 \end{cases}$$

where

$$\begin{cases} Y^{\nu,h}_{t_n,y}(t_n) = y, \\ Y^{\nu,h}_{t_n,y}(t_{i+1}) = Y^{\nu,h}_{t_n,y}(t_i)\Big(1 - r_i h - \tilde{g}(t_i, \nu_i)h + \sigma_i^{-1}(r_i - b_i - \nu_i)\sqrt{h}\zeta_i\Big) \end{cases}$$

for $i = n, \ldots, N-1$. One has

$$\tilde{V}(t_n, y) = \inf_{\nu \in \mathcal{V}_h} \mathbb{E}\Big[\tilde{U}(Y^{\nu,h}_{t_n,y}(T))\Big].$$

PDE-based error estimates

For the scheme we described: given a smooth function f

$$\left|\frac{1}{h}\Big(f(t_{n-1},x)-\sup_{a\in A}\mathcal{S}[f(t_n,\cdot)](t_n,x,a)\Big)-\Big(-\partial_t f+H(t_n,x,\partial_x f,\partial_{xx} f)\Big)\right|$$

$$\leq Kh\Big(\|\partial_t^2 f\|_\infty+\|\partial_x f\|_\infty+\|\partial_x^3 f\|_\infty+\|\partial_x^4 f\|_\infty\Big) \quad \text{(consistency error).}$$

Let $\varepsilon > 0$ be a regularization parameter.
This would give (Debrabant-Jakobsen '13):

$$\|V-v\|_\infty \leq \underbrace{Kh\varepsilon^{-3}}_{\text{consistency error}} + \underbrace{K\varepsilon}_{\text{regularization error}}$$

$$= Kh^{1/4} \qquad \text{for } \varepsilon = h^{1/4}.$$

Direct error estimates

The Euler-Maruyama approximation gives:

$$\begin{aligned}\left|v_h(t_n,x) - V_{EM}(t_n,x)\right| &\leq \sup_{\alpha\in\mathcal{A}_h}\left|\mathbb{E}[U(X^{\alpha,EM}_{t_n,x}(T))] - \mathbb{E}[U(X^{\alpha}_{t_n,x}(T))]\right| \\ &\leq L_U \sup_{\alpha\in\mathcal{A}_h}\left|\mathbb{E}[X^{\alpha,EM}_{t_n,x}(T)] - \mathbb{E}[X^{\alpha}_{t_n,x}(T)]\right| \\ &\leq K(1+|x|)h^{1/2}.\end{aligned}$$

For any smooth function f, $a \in A$, the Gauss-Hermite approximation gives at any step:

$$\left|\ \mathbb{E}\left[f(X^{a,EM}_{t_n,x}(t_{n+1}))\right] - \mathbb{E}\left[f(X^{a,h}_{t_n,x}(t_{n+1}))\right]\ \right| \leq K\|f^{(2M)}\|_\infty x^{2M} h^M.$$

Direct error estimates: lower bound

Let ε be a regularization parameter and $K \equiv K(M, x)$.

- Perturbed problem:

$$V^{\varepsilon}(t_n, x) := \sup_{\alpha \in \mathcal{A}_h, e \in \mathcal{E}_h} \mathbb{E}\Big[U(X_{t_n,x}^{\alpha,e,EM}(T))\Big].$$

  One has:

$$\sup_{n=0\ldots N} \Big| V_{EM}(t_n, x) - V^{\varepsilon}(t_n, x) \Big| \leq K\varepsilon.$$

- Regularization:

$$V_{\varepsilon}(t_n, x) := \int_{\mathbb{R}} V^{\varepsilon}(t_n, x - y) \mu_{\varepsilon}(y) dy.$$

  One has:
  - $\sup_{n=0\ldots N} \Big| V_{\varepsilon}(t_n, x) - V^{\varepsilon}(t_n, x) \Big| \leq K\varepsilon$;
  - $\|\partial_x^i V_{\varepsilon}\|_{\infty} \leq K\varepsilon^{i-1}$;
  - $V_{\varepsilon}(t_n, x) \geq \sup_{a \in A} \mathbb{E}\Big[V_{\varepsilon}(t_{n+1}, X_{t_n,x}^{a,EM}(t_{n+1}))\Big].$

Direct error estimates: lower bound

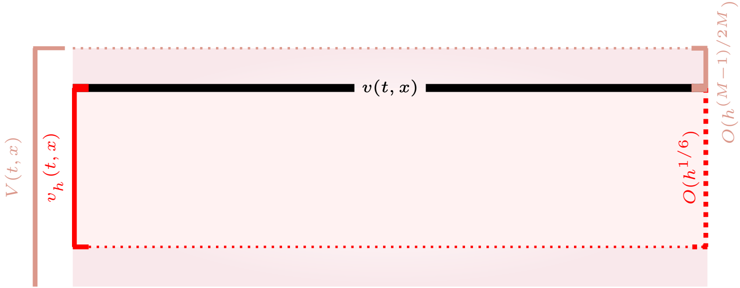
One has

$$\sup_{n=0,\dots,N}\Big(V(t_n,x)-v_h(t_n,x)\Big) \le \underbrace{Kh^{M-1}\varepsilon^{1-2M}}_{\substack{\text{Gauss-Hermite error}\\(V-V_\varepsilon)}} + \underbrace{K\varepsilon}_{\substack{\text{regularization error}\\(V_\varepsilon-V_{EM})}} + \underbrace{Kh^{1/2}}_{\substack{\text{EM error}\\(V_{EM}-v_h)}}$$

$$\le Kh^{(M-1)/2M} \qquad \text{for } \varepsilon = h^{(M-1)/2M}$$

and, using Krylov ('99),

$$0 \le v(t,x) - v_h(t,x) \le Kh^{1/6}.$$



Direct error estimates: upper bound

For the dual problem we also have

$$\tilde{v}(t_n, y) - \tilde{v}_h(t_n, x) \leq 0$$

and for $p = \frac{M-1}{2M}$

$$\sup_{n=0,\dots,N} \left(\tilde{v}_h(t_n, x) - \tilde{V}(t_n, x)\right) \leq K h^p.$$

Therefore

$$\begin{aligned} v(t_n, x) &= \inf_{y \geq 0} \left\{\tilde{v}(t_n, y) + xy\right\} \\ &\leq \inf_{y \geq 0} \left\{\tilde{V}(t_n, y) + K(M, y) h^p + xy\right\} \\ &\leq \inf_{y \geq 0} \left\{\tilde{V}(t_n, y) + xy\right\} + \tilde{K}(M, x) h^p. \end{aligned}$$

Direct error estimates: conclusions

We can conclude that there exists some $C \equiv C(M, x)$ such that

$$-Ch^p \leq v(t_n, x) - V(t_n, x) \leq \inf_{y \geq 0} \left\{ \tilde{V}(t_n, y) + xy \right\} - V(t_n, x) + Ch^p$$

for $p = \frac{M-1}{2M}$, for all $n = 0, \ldots, N$.

Remarks:

- If $M = 2$ then $p = h^{1/4}$;
- For $M \to \infty$ one has $p \to 1/2$.

Future work

Future direction: a priori bounds

Can some duality relation still be found between V and $\tilde{V}$?
Let $\mathcal{X}(t_n) = \bigcup_{x\geq 0} \mathcal{X}(t_n, x)$ where

$$\mathcal{X}(t_n, x) := \left\{ f \ \mathcal{F}_T\text{-meas.} : \exists \alpha \in \mathcal{A} \text{ s.t. } \quad 0 \leq f \leq X^{\alpha}_{t_n,x}(T) \text{ a.s.} \right\}$$

(analogous definitions for $\mathcal{Y}(t_n), \mathcal{X}^h(t_n)$ and $\mathcal{Y}^h(t_n)$).
Necessary assumptions (Kramkov-Schachermayer '99):

(X) $\mathcal{X}(t_n, x)$ convex;
$\mathcal{X}(t_n, \lambda x) = \lambda \mathcal{X}(t_n, x) \quad \forall \lambda > 0$;
$\mathbb{1} \in \mathcal{X}(t_n)$;
g $\mathcal{F}_T$-meas: $0 \leq g \leq f \in \mathcal{X}(t_n, x)$ implies $g \in \mathcal{X}(t_n, x)$;

(Y) $\mathcal{Y}(t_n, y)$ closed and convex;

(XY) for all $f \in \mathcal{X}(t_n)$ and $y \geq 0$: $\sup_{\ell \in \mathcal{Y}(t_n,y)} \mathbb{E}[f\ell] = \inf_{\substack{x\geq 0: \\ f \in \mathcal{X}(t_n,x)}} \{xy\}$.

Future direction: a priori bounds

Polarity property between $X^{\alpha}_{t,x}$ and $Y^{\nu}_{t,y}$ (continuous time):

- $\mathbb{E}\Big[X^{\alpha}_{t,x}(T)Y^{\nu}_{t,y}(T)\Big] - xy \leq 0;$
- $\mathbb{E}\Big[X^{\alpha}_{t,x}(T)Y^{\nu}_{t,y}(T)\Big] - xy = 0 \quad \text{if} \quad \tilde{g}(t,\nu_t) = g(t,\alpha_t) - \alpha_t\nu_t, \forall t.$

Let h be small enough so that $X^{\alpha,h}_{t,x}, Y^{\nu,h}_{t_n,y} \geq 0$ (ok if A and Γ bounded).
Straightforward calculations show an approximate polarity property between $X^{\alpha,h}_{t_n,x}$ and $Y^{\nu,h}_{t_n,y}$:

- $\mathbb{E}\Big[X^{\alpha,h}_{t_n,x}(T)Y^{\nu,h}_{t_n,y}(T)\Big] - xy \leq Chxy;$
- $\mathbb{E}\Big[X^{\alpha,h}_{t_n,x}(T)Y^{\nu,h}_{t_n,y}(T)\Big] - xy \geq -Chxy \quad \text{if} \quad \tilde{g}(t_i,\nu_i) = g(t_i,\alpha_i) - \alpha_i\nu_i, \ \forall i$

for some constant $C \geq 0$ (independent of α, ν).

Idea: Try to obtain $\inf_{y\geq 0}\Big\{\tilde{V}(t_n,y) + xy\Big\} - V(t_n,x) \leq K(x,M)h.$

Extensions

- Consider a investment/consumption problem:

$$dX(s) = X(s)\Big(r_s + \alpha_s(b_s - r_s)\Big)ds + \alpha_s\sigma_s X(s)d\mathcal{B}_s - c_s ds$$

$$v(t,x) = \sup_{\alpha,c} \mathbb{E}\Big[U(X^{\alpha}_{t,x}(T)) + \int_t^T U(s,c_s)ds\Big].$$

- Take into account higher order schemes (Milstein, etc.).

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Thank you.