Martingale Optimal Transport

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We are given

- two closed, convex sets $X, Y \subset \mathbb{R}^d$,
- two probability measures μ, ν defined on Borel sets $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively.

A transport map is a function $T : X \to Y$, satisfying

$$\nu = T_{\#}\mu, \quad \text{i.e.,} \quad \nu(A) = \mu\left(\{x \in X \ : \ T(x) \in A\}\right), \ \ \forall A \in \mathcal{B}(X).$$

In general, there may be no such maps.

Given the above structure and a reward function

$$\xi : \Omega := X \times Y \to \mathbb{R},$$

the optimal transport problem is to find the transport T that maximizes

$$J(T) := \int_X \xi\,(x, T(x))\;\mu(dx).$$

This is a hard *nonlinear* problem!

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For a transport map, define a probability measure on Ω by

$$\mathbb{P}(C) = \mu\left(\{x \in X \,:\, (x, T(x)) \in C\}\right), \quad C \in \mathcal{B}(\Omega).$$

Then, the definition of $\mathbb{P}$ and $\nu = T_{\#}\mu$ imply that

$$\mathbb{P}(A \times Y) = \mu(A),\;\; A \in \mathcal{B}(X), \quad \text{and} \quad \mathbb{P}(X \times B) = \nu(B),\;\; B \in \mathcal{B}(Y).$$

Let $\mathcal{M}(\mu, \nu)$ is the set of all probability measures $\mathbb{P}$ defined on Borel sets $\mathcal{B}(\Omega)$ satisfying the above marginal constraints.

- For a given transport map T there is a generalized transport $\mathbb{P}$. Indeed, define $\mathbb{P}$ by

$$\mathbb{P}(dx, dy) = \delta_{T(x)}(dy)\ \mu(dy).$$

Then, $\mathbb{P} \in \mathcal{M}(\mu, \nu)$.

- However, the converse is not true. There are many more generalized transports.
- Also, $\mathcal{M}(\mu, \nu)$ is always nonempty.

$$\mu \times \nu \in \mathcal{M}(\mu, \nu).$$

The generalized map corresponds to the transport map T is

$$\mathbb{P}(dx, dy) = \delta_{T(x)}(dy)\ \mu(dy).$$

The objective functional is

$$\begin{aligned} J(\mathbb{P}) &= \int_X \xi(x, T(x))\ \mu(dx) \\ &= \int_\Omega \xi(x, y)\ \delta_{T(x)}(dy)\ \mu(dx) \\ &= \int_\Omega \xi(x, y)\ \mathbb{P}(dx, dy) \\ &= \mathbb{E}_{\mathbb{P}}[\xi]. \end{aligned}$$

The relaxation of the Monge problem is

$$\mathcal{P}(\xi) := \sup_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \mathbb{E}_{\mathbb{P}}[\xi].$$

- This is a *linear* problem and there always exists an optimizer.
- We are given the marginals of the measure $\mathbb{P}$ and the optimal transport problem is to find the "best" correlation structure.
- It has many natural applications in risk management.

For $h \in \mathcal{L}^1(X, \mu)$, $g \in \mathcal{L}^1(Y, \nu)$, set

$$\begin{aligned} (h \oplus g)(z) &:= h(x) + g(y), \quad z = (x, y) \in \Omega = X \times Y, \\ \mathcal{L}(h, g) &:= \int_X h d\mu + \int_Y g d\nu. \end{aligned}$$

Then, for every $\mathbb{P} \in \mathcal{M}(\mu, \nu)$,

$$\mathbb{E}_{\mathbb{P}}[h \oplus g] = \mathcal{L}(h, g).$$

This suggests that the dual is given by

$$\boxed{\mathcal{D}(\xi) := \inf \{ \mathcal{L}(h, g) \ : \ h \oplus g \geq \xi \} .}$$

Consider the subspace of $C_b(\Omega)$,

$$\mathcal{A} := \{\, f \in C_b(\Omega) \ : \exists\, g, h \text{ such that } f = h \oplus g - \mathcal{L}(h, g) \,\}.$$

Then, $\mathcal{M}(\mu, \nu)$ is the set of all Radon measures that annihilate the set $\mathcal{A}$, i.e.

$$\mathbb{P} \in \mathcal{M}(\mu, \nu), \quad \Leftrightarrow \quad \mathbb{E}_{\mathbb{P}}[\, f\,] = 0, \ \forall\, f \in \mathcal{A}.$$

This is the reason behind the duality.

Theorem (Kantorovich 1940, Kellerer 1984)

For any bounded, Borel measurable ξ,

$$\mathcal{P}(\xi) = \mathcal{D}(\xi).$$

- This follows immediately from Fenchel-Moreau theorem for continuous functions ξ.
- A straigtforward approximation argument proves it for all upper-semicontinuous ξ's.
- General result is due to Kellerer.
- The inequality $\mathcal{D} \geq \mathcal{P}$ is easy.

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Suppose $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$. Then, the primal problem (in Kantorovich formulation) is

$$\text{maximize} \sum_{i,j} \xi(x_i, y_j) p_{ij}$$

over all p_{ij}'s satisfying

$$\mu_i = \sum_j p_{ij}, \quad i = 1, \ldots, n,$$

$$\nu_j = \sum_i p_{ij}, \quad j = 1, \ldots, m,$$

$$p_{ij} \geq 0, \quad i = 1, \ldots, n, \; j = 1, \ldots, m.$$

Let $h = (h_1, \ldots, h_n)$ be the dual variables for the constraint :

$$\mu_i = \sum_j p_{ij}, \quad i = 1, \ldots, n,$$

Let $g = (h_1, \ldots, h_m)$ be the dual variables for the constraint :

$$\nu_j = \sum_i p_{ij}, \quad j = 1, \ldots, m.$$

Then, the classical duality for linear program implies that the dual objective is to minimize

$$\sum_i h_i \, \mu_i + \sum_j g_j \, \nu_j.$$

Since the primal variables $p_{ij} \geq 0$, the dual constraint is

$$h_i + g_j \geq \xi(x_i, y_j), \quad \forall \; i, j.$$

A generalized transport q_{ij} is a regular transport map if

$$q_{ij} = \begin{cases} \mu_i & \text{if } j = T(i), \\ 0 & \text{if } j \neq T(i). \end{cases}$$

Then, T is a transport map if

$$\nu_j = \sum_{\{i \,:\, T(i)=j\}} \mu_j.$$

This is a difficult constraint and may not be possible. However, the generalized transport always exists :

$$\text{set } \ q_{ij} = \mu_i \, \nu_j, \qquad \text{then} \qquad q \in \mathcal{M}(\mu, \nu).$$

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We restrict the class of generalized transports to be martingale as well. Indeed, let $\mathcal{Q}(\mu, \nu)$ be the set of all $\mathbb{Q} \in \mathcal{M}(\mu, \nu)$ satisfying

$$\mathbb{E}_{\mathbb{Q}}\left[\gamma(x) \cdot (y - x)\right] = 0,$$

for all bounded, Borel measurable γ. Strassen (1965) proved that $\mathcal{Q}(\mu, \nu)$ is non-empty if and only if μ and ν are in convex order.

The primal problem is defined as

$$\boxed{\mathcal{P}(\xi) := \sup_{\mathbb{Q} \in \mathcal{Q}(\mu,\nu)} \mathbb{E}_{\mathbb{Q}}[\xi].}$$

Suppose there exists $\mathbb{Q} \in \mathcal{Q}(\mu, \nu)$ and $\phi : \mathbb{R}^d \to R$ is convex. Recall that $X, Y \subset \mathbb{R}^d$. By conditioning (and abuse of notation)

$$\mathbb{E}_{\mathbb{Q}}[\phi(y)] = \mathbb{E}_{\mathbb{Q}}[\ \mathbb{E}_{\mathbb{Q}}[\phi(y) \mid x]\ ].$$

Then, by Jensen's inequality and the fact that $\mathbb{P}$ is a martingale measure,

$$\mathbb{E}_{\mathbb{Q}}[\phi(y) \mid x] \geq \phi\left(\mathbb{E}_{\mathbb{Q}}[\ y \mid x]\right) = \phi(x).$$

Combining and using marginals,

$$\int_Y \phi d\nu = \mathbb{E}_{\mathbb{Q}}[\phi(y)] \geq \mathbb{E}_{\mathbb{Q}}[\phi(x)] = \int_X \phi d\mu.$$

So μ and ν must be in convex order.

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Marginal constraints imply that for every $\mathbb{Q} \in \mathcal{Q}(\mu, \nu)$,

$$\mathbb{E}_{\mathbb{Q}}[h \oplus g] = \mathcal{L}(h, g) = \int h d\mu + \int g d\nu.$$

Martingale condition is

$$\mathbb{E}_{\mathbb{Q}}\left[\gamma(x) \cdot (y - x)\right] = 0, \quad \gamma \in \mathcal{L}^{\infty}(X).$$

Combining all above we conclude that for all $\mathbb{Q} \in \mathcal{Q}(\mu, \nu)$,

$$\mathbb{E}_{\mathbb{Q}}\left[H(h, g, \gamma)\right] = \mathcal{L}(h, g)$$

where

$$H(h, g, \gamma)(z) = (h \oplus g)(z) + \gamma(x) \cdot (y - x).$$

$$\mathbb{E}_{\mathbb{Q}}\left[H(h,g,\gamma)\right] = \mathcal{L}(h,g), \quad \forall\, \mathbb{Q} \in \mathcal{Q}(\mu,\nu),$$

$$H(h,g,\gamma)(z) = h(x) + g(y) + \gamma(x)\cdot(y-x).$$

This suggests that the dual is

$$\mathcal{D}(\xi) := \inf\left\{\mathcal{L}(h,g) \;:\; H(h,g,\gamma) \geq \xi\right\}.$$

The main difference is the new term $\gamma(x)\cdot(y-x)$. This is due to the martingale constraint imposed on the probability measures $\mathbb{Q}$.

Theorem

Suppose μ and ν are in convex order, i.e.,

$$\int \phi d\mu \leq \int \phi d\nu,$$

for every convex ϕ. Then,

$$\mathcal{P}(\xi) = \mathcal{D}(\xi),$$

for every bounded and continuous ξ.

This follows directly from the Fenchel-Moreau theorem.

Let $C_b(\Omega)$ be the set of all bounded, continuous functions. Then,

$$\mathcal{D} : C_b(\Omega) \to \mathbb{R}.$$

It is clear that

- $\mathcal{D}$ is convex ;
- $\mathcal{D}$ is Lipschitz in the sup-norm ;
- For any $\lambda > 0$ and $\xi \in C_b(\Omega)$, $\mathcal{D}(\lambda\xi) = \lambda\mathcal{D}(\xi)$.
- $$|\mathcal{D}(\xi)| \leq \|\xi\|_\infty.$$

So $\mathcal{D}$ is a proper convex function and Fenchel-Moreau theorem applies.

Fenchel-Moreau imples that,

$$\mathcal{D}(\xi) = \sup_{\mathbb{Q}\in(C_b(\Omega))^*} \{\, \mathbb{Q}(\xi) - \mathcal{D}^*(\mathbb{Q})\},$$

$$\mathcal{D}^*(\mathbb{Q}) = \sup_{\xi\in C_b(\Omega)} \{\, \mathbb{Q}(\xi) - \mathcal{D}(\xi)\}.$$

- $\mathcal{D}^*(\mathbb{Q}) = 0$ if $\mathbb{Q}$ satisfies $\mathbb{Q}(H(h, g, \gamma)) = 0$ for all h, g, γ. And $\mathcal{D}^*(\mathbb{Q}) = +\infty$, otherwise. (will be discussed later).
- Marginal constraints imply that any $\mathbb{Q} \in (C_b(\Omega))^*$ satisfying above is a probability measure.
- Hence $\mathcal{D}^*(\mathbb{Q}) = 0$ if and only if $\mathbb{Q} \in \mathcal{Q}(\mu, \nu)$.

$$\mathcal{D}^*(\mathbb{Q}) = \sup_{\xi \in C_b(\Omega)} \{ \mathbb{Q}(\xi) - \mathcal{D}(\xi) \}.$$

- $\mathcal{D}(0) = 0$ implies that $\mathcal{D}^* \geq 0$.
- If $\mathbb{Q}(\xi) > \mathcal{D}(\xi)$ for some ξ, by scaling we conclude that $\mathcal{D}^*(\mathbb{Q}) = +\infty$.
- Hence, $\mathcal{D}^*(\mathbb{Q}) = 0$ iff $\mathbb{Q}(\xi) \leq \mathcal{D}(\xi)$ for every ξ. And the definition of $\mathcal{D}$ imply that $\mathbb{Q}(H(h, g, \gamma)) = 0$.

Duality does not hold for general ξ. Indeed, consider the example :

- $X = Y = [0, 1]$,
- μ and ν are the Lebesgue measures.
- Then, $\mathcal{Q}(\mu, \nu) = \{\mathbb{Q}^*\}$ where $\mathbb{Q}^*$ is the uniform distribution on the diagonal ; $T(x) = x$ is the only transport map.
- Set ξ^* to be zero on the diagonal and one off the diagonal.
- It is clear that $\mathcal{P}(\xi^*) = 0$.
- However, $\mathcal{D}(\xi^*) = 1$.

We modify the dual slightly by requiring the inequality to hold quasi-surely (q.s., in short) :

We say that $\eta \geq \xi$, $\mathcal{Q}$-quasi surely if

$$\mathbb{Q}\left(\{\eta < \xi\}\right) = 0, \quad \forall\, \mathbb{Q} \in \mathcal{Q}(\mu, \nu).$$

Then, we define the dual by

$$\hat{\mathcal{D}}(\xi) := \inf \left\{ \mathcal{L}(h, g) \ : \ H(h, g, \gamma) \geq \xi, \ \mathcal{Q} - q.s. \right\}.$$

Theorem (Beiglböck, Nutz & Touzi, 2015)

$$\mathcal{P}(\xi) = \hat{\mathcal{D}}(\xi), \quad \forall\, \xi \in \mathcal{L}^{\infty}(\Omega).$$

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- Trading is only allowed in two future time points, $0 < t_1 < t_2$;
- We can dynamically trade the stock whose future values will be $\mathbb{S}_1,\ \mathbb{S}_2$;
- No assumptions on $\mathbb{S}_1,\ \mathbb{S}_2$ are made but $\mathbb{S}_0$ is given;
- We can also now buy any option of the form $h(\mathbb{S}_1)$ and $h(\mathbb{S}_2)$ for a known price of $\int h d\mu$, respectively $\int g d\nu$. Here g, h are arbitrary and can be chosen by the investor. But probability measures ν and μ are given.
- We want to super-replicate a claim $\xi(\mathbb{S}_1, \mathbb{S}_2)$.

Mathematically, we want to minimize (with zero interest rate) the prices of the options $h(\mathbb{S}_1)$ and $g(\mathbb{S}_2)$, i.e.,

$$\text{minimize} \left[\int h(\mathbb{S}_1)\, \mu(d\mathbb{S}_1) + \int g(\mathbb{S}_2)\, \nu(d\mathbb{S}_2) \right],$$

over all options (g, h) that together with a dynamic trading strategy θ_0, $\theta_1(\mathbb{S}_1)$, dominate the claim $\xi(\mathbb{S}_1, \mathbb{S}_2)$.

θ_0 = shares held today,

$\theta_1(\mathbb{S}_1)$ = shares will be bought at time t_1, and this decision will depend on the value of the stock at time t_1.

More compactly (with $\mathbb{S}_0$ given),

$$\text{minimize} \left[\int h \; \mu + \int g \; \nu \right],$$

over all $h \in \mathbb{L}^1(\mathbb{R}^d, \mu)$, $g \in \mathbb{L}^1(\mathbb{R}^d, \nu)$ so that there exists $\theta_0 \in \mathbb{R}^d$, $\theta_1 : \mathbb{R}^d \to \mathbb{R}^d$ satisfying,

$$\theta_0 \cdot (\mathbb{S}_1 - \mathbb{S}_0) + \theta_1(\mathbb{S}_1) \cdot (\mathbb{S}_2 - \mathbb{S}_1) + h(\mathbb{S}_1) + g(\mathbb{S}_2) \geq \xi(\mathbb{S}_1, \mathbb{S}_2),$$

for all $\mathbb{S}_1, \; \mathbb{S}_2 \in \mathbb{R}^d$.

We make the identification,

$$x = \mathbb{S}_1, \quad y = \mathbb{S}_2, \quad H(x) = h(x) + \theta_0 \cdot (x - \mathbb{S}_0), \quad G(y) = g(y).$$

The problem is to minimize $\int H d\mu + \int G d\nu$, over all (G, H) so that

$$H(x) + G(y) + \theta_1(x) \cdot (y - x) \geq \xi(x, y),$$

for some θ_1, for all (x, y). The term $\theta_1(x) \cdot (y - x)$ in the duality constraint is new.

The dual is given by

$$\text{maximize} \;\; \mathbb{E}_{\mathbb{Q}}\left[c(\mathbb{S}_1, \mathbb{S}_0)\right],$$

over all martingale measures $\mathbb{Q} \in \mathcal{Q}(\nu, \mu)$, i.e, for $\Omega := \mathbb{R}^2$, the canonical process

$$\mathbb{S}_0((\omega_1, \omega_2)) = \mathbb{S}_0, \quad \mathbb{S}_1((\omega_1, \omega_2)) = \omega_1, \quad \mathbb{S}_2((\omega_1, \omega_2)) = \omega_2,$$

with the canonical filtration is a $\mathbb{Q}$ martingale, i.e.,

$$\mathbb{E}\left[\,\mathbb{S}_2 \mid \mathbb{S}_1\right] = \mathbb{S}_1.$$

Assumptions are $\int y\mu(dy) = \int x\nu(dx) = \mathbb{S}_0$ and that they are in convex order. So that the set of martingale measures satisfying the constraint is non-empty.

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Suppose that for some G, H,

$$H(\mathbb{S}_1) + G(\mathbb{S}_2) + \theta_1(\mathbb{S}_1) \cdot (\mathbb{S}_2 - \mathbb{S}_1) \geq \xi(\mathbb{S}_1, \mathbb{S}_2),$$

for some θ_1, and for all $(\mathbb{S}_1, \mathbb{S}_2)$. Let $\mathbb{Q} \in \mathcal{Q}(\nu, \mu)$ be a martingale measure. Then,

$$\mathbb{E}_Q\left[\theta_1(\mathbb{S}_1) \cdot (\mathbb{S}_2 - \mathbb{S}_1)\right] = 0,$$

and the marginals imply that

$$\mathbb{E}_{\mathbb{Q}}\left[H(\mathbb{S}_1)\right] = \int H d\mu, \qquad \mathbb{E}_{\mathbb{Q}}\left[G(\mathbb{S}_2)\right] = \int G d\nu.$$

So if we integrate the first inequality with respect to $\mathbb{Q}$,

$$\int H d\mu + \int G d\nu \geq \mathbb{E}_{\mathbb{Q}}\left[\xi(\mathbb{S}_1, \mathbb{S}_2)\right].$$

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Suppose that all right-continuous processes with left limits (namely *càdlàg* processes) are possible stock price processes :

$$\Omega := \mathbb{D} := \{\mathbb{S} : [0,T] \to \mathbb{R}_+^d \mid \mathbb{S}_0 = (1,\dots,1),\ \textit{càdlàg}\ \}.$$

All European options $g(\mathbb{S}_T)$ maturing only at the final time are traded. Its price is assumed to be

$$\mathcal{L}(g) = \int_{\mathbb{R}_+^d} g d\mu, \qquad \forall\, g \in \mathbb{L}^1(\mathbb{R}_+, \mu).$$

Technically, we assume that

$$\int_{\mathbb{R}^+} x_k \mu(dx) = 1, \quad \forall\, k = 1,\dots,d.$$

This allows for martingale measures.

A general European path-dependent claim, ξ, is given,

$$\xi = G(\mathbb{S}), \quad \text{where } G : \mathbb{D} \to \mathbb{R}.$$

The minimal super-replication cost also provides the upper bound of the price interval and is given by

$$\mathcal{D}(G) \quad := \quad \inf \left\{ \int g d\mu \;\middle|\; \exists \gamma \text{ admissible such that} \right.$$
$$\left. \int_{[0,T]} \gamma_t(\mathbb{S}) d\mathbb{S}_t + g(\mathbb{S}_T) \geq G(\mathbb{S}),\ \forall\, \mathbb{S} \in \mathbb{D} \right\}.$$

Clearly, admissibility restrictions on γ are needed, such as a lower bound, predictability.

Dual refers to a probabilistic structure, which we now introduce. $\Omega := \mathbb{D}$ is as before and let $\mathbb{S}$ be the canonical process and $\mathcal{F}_t$ be the canonical filtration.

As before $\mathcal{Q}(\mu)$ is the set of all martingale measures such that the probability distribution of $\mathbb{S}_T$ under $\mathbb{Q}$ is μ, i.e.,

$$\mathbb{Q}\left(\mathbb{S}_T \in B\right) = \mu(B).$$

Theorem (DS : continuous (2012) PTRF - *càdlàg* (2015) SPA)

Assume that G is bounded and uniformly continuous with respect to the Skorokhod metric. Then,

$$\mathcal{D}(G) = \sup_{\mathbb{Q} \in \mathcal{Q}(\mu)} \mathbb{E}_{\mathbb{Q}}[G(\mathbb{S})].$$

We could also consider finitely many in between marginals.

Continuity of the claim G is essential. For general claims, one needs the quasi-sure set-up that will be discussed later.

- Previous results fix a probability measure $\mathbb{P}$ and the inequalities are understood $\mathbb{P}$-a.s.
- In the robust case, we fix the marginal μ, but otherwise have no dominating measure. Inequalities are pointwise.
- There is also an in between case. We fix a class of probability measures $\mathcal{P}$. Then, the inequalities are to be understood $\mathcal{P}$ quasi-surely.
- In discrete time Bouchard & Nutz (2014) proved the duality for general claims (not only continuous) and the fundamental theorem of asset pricing in this structure.
- Continuous time quasi-sure models are studied by Biagini, Bouchard, Kardaras & Nutz with continuous paths.

References - Formulation

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FTAP= Fundamental Theorem of Asset Pricing.
MOT= Martingale Optimal Transport
SEP= Skorokhod Embedding Problem.

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- Optimal transport can be used in risk management context to look for the optimal correlation structure.
- Martingale restriction is useful in applications where the marginals are the distributions at different time points.
- One may imagine restrictions other than martingality.

THANK YOU FOR YOUR ATTENTION.