Stochastic Control of Path-Dependent Systems, Application to Principal-Agent Problem

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Outline

1. The Principal-Agent Problem
   - Formulation
   - A sub-optimal Principal problem
   - Main Result
2. Solving the representation problem
   - Case of un-controlled diffusion
   - General controlled diffusion case

Output process

Output : controlled state process in $\mathbb{R}^d$: weak solution $\mathbb{P}$ of

$$\text{(SDE)} \qquad dX_t = \sigma_t(X, \beta_t)\big[\lambda_t(X, \alpha_t)dt + dW_t\big)$$

- W is a Brownian motion with values in $\mathbb{R}^n$
- Effort=Control process $\nu^{\mathbb{P}} = (\alpha^{\mathbb{P}}, \beta^{\mathbb{P}})$ with values in $A \times B$

Hidden effort $\equiv$ Moral hazard

- Observation of X does not give access to the drift $\sigma\lambda$
- Observation of X gives access to $\sigma\sigma^\top$ but not to σ

Agent's problem

Agent solves the following control problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}\in\mathcal{P}} J^A(\mathbb{P},\xi); \quad J^A(\mathbb{P},\xi) := \mathbb{E}^{\mathbb{P}}\Big[\underbrace{K_T^{\mathbb{P}}\xi}_{\text{contract}} - \int_0^T K_t^{\mathbb{P}} \underbrace{c_t(\nu_t^{\mathbb{P}})}_{\text{cost}} dt\Big]$$

where $K_t^{\mathbb{P}} := e^{-\int_0^t k_s(\nu_s^{\mathbb{P}})ds}$, $\xi = \xi(X_{\wedge T})$, and

$$dX_t = \sigma_t(\beta_t^{\mathbb{P}})\big[\lambda_t(\alpha_t^{\mathbb{P}})dt + dW_t^{\mathbb{P}}\big), \quad \mathbb{P} - \text{a.s.}$$

Contract : compensation for the management of X

$\mathbb{P}^*(\xi)$: solution of $V^A(\xi)$, if exists

k and c may also be path-dependent

Comments on path-depency

Path-dependency of ξ is crucial : Non-Markov stochastic control

One of our main objective is

Path-dependent stochastic control should be addressed exactly as Markovian stochastic control

• Well-know for Pontryagin Maximum Principle Approach

• Dynamic programming principle also available in path-dependent control problems

• Dynamic programming equation (HJB) : Backward SDEs and Second order extension, viscosity solutions...

The Principal problem

Moral hazard : Principal chooses $\xi(X)$ based on X only

Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} J^P(\mathbb{P}^\star(\xi), \xi) \quad \text{where} \quad J^P(\mathbb{P}, \xi) := \mathbb{E}^{\mathbb{P}}\Big[K_T^P U(\ell - \xi)\Big]$$

where $K_T^P := e^{-\int_0^T k_t^P(X)dt}$, and

- $\mathbb{P}^*(\xi)$: solution of Agent problem given the contract ξ
- $\ell := \ell(X_{\wedge T})$, maps Ω to $\mathbb{R}$: liquidation function
- Participation constraint :

$$\Xi_R := \big\{\xi \in \mathcal{F}_T^X : \mathbb{P}^*(\xi) \text{ exists, integrability, and } V_0^A(\xi) \geq R\big\}$$

Moral hazard in insurance

- Fully insured drivers do not have incentive to increase caution

- Health insurance misleadingly lowers out-of-pocket expenses, and thus increases demand for medical services

- Un-employment insurance may give negative incentive to return to the job market

Regulation of Financial Sector

How to design financial regulation so as fund managers have the incentive to care about outside damage ?

Existing literature

• Moral hazard was introduced by Adam Smith (1723-1790)

• Principle / Agent problem : Non-zero sum Stackelberg game, Agent plays first

• Continuous-time formulation : Holstrom & Milgrom '87 (Econometrica), ..., Sannikov '08 : un-controlled diffusion, all based on cases of explicit solutions, un-controlled diffusion

• Cvitanić & Zhang '13 : calculus of variations $\Longrightarrow$ Pontryagin Maximum Principle leading to a system of Forward-Backward SDEs... restricted to un-controlled diffusion

Our objective

- Simple and systematic solution by standard dynamic programming (inspired from Sannikov '08)

- Include diffusion control

- Dimension of X does not matter,

- no need to explicit solution, access to numerical approximation

- Possibility includes additional features, e.g. limited liability

Summarizing Principal-Agent problem

- Agent solves the control problem :

$$V_0^A(\xi) := \sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\Big[K_T^{\mathbb{P}}\xi - \int_0^T K_t^{\mathbb{P}} c_t(\nu_t^{\mathbb{P}})dt\Big]$$

where the Output process is a weak solution $\mathbb{P}$ of the SDE

$$dX = \sigma_t(X, \beta_t)\big[\lambda_t(X, \alpha_t)dt + dW_t\big]$$

- Principal solves the optimization problem

$$V_0^P := \sup_{\xi \in \Xi_R} \mathbb{E}^{\mathbb{P}^\star(\xi)}\Big[K_T^P U(\ell - \xi)\Big]$$

where Ξ_R : $\xi \in \mathcal{F}_T^X$, such that $V_0^A(\xi) \geq R$

Path-dependent Hamiltonian

- Path-dependent Hamiltonian for the Agent problem :

$$H_t(\omega, y, z, \gamma) \quad := \quad \sup_{a,b} \Big\{ \sigma_t(\omega,a)\lambda_t(\omega,b)\cdot z + \tfrac{1}{2}\sigma_t\sigma_t^\top(\omega,a) : \gamma \\ - k_t(\omega,a,b)y - c_t(\omega,a,b) \Big\}$$

- For $Y_0 \in \mathbb{R}$ and Z, Γ $\mathbb{F}^X -$ prog meas, define

$$Y_t^{Z,\Gamma} = Y_0 + \int_0^t Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

$$\mathbb{P} - \text{a.s. for all } \mathbb{P} \in \mathcal{P}$$

A class of revealing conracts

Proposition $V_0^A(Y_T^{Z,\Gamma}) = Y_0$ and any maximizer of the Hamiltonian $(a^\star, b^\star)(Y, Z, \Gamma)$ induces a solution $\hat{\mathbb{P}}^{Z,\Gamma}$ of the Agent problem

$$dX_t = \nabla_z H_t(X, Y_t, Z_t, \Gamma_t)dt + 2\nabla_\gamma H_t(X, Y_t, Z_t, \Gamma_t)dW_t, \ \hat{\mathbb{P}}^{Z,\Gamma}\text{−a.s.}$$

Sub-optimal stochastic control problem

Induces lower bound $V_0^P \geq \sup_{Y_0 \geq R} \underline{V}_0(X_0, Y_0)$, where :

$$\underline{V}_0(X_0, Y_0) := \sup_{Z,\Gamma} \mathbb{E}^{\widehat{\mathbb{P}}^{Z,\Gamma}}\Big[U\big(\ell(X_T) - Y_T^{Z,\Gamma}\big)\Big]$$

where the dynamics under $\widehat{\mathbb{P}}^{Z,\Gamma}$ are

$$dX_t = \nabla_z H_t(X, Y_t, Z_t, \Gamma_t)dt + 2\nabla_\gamma H_t(X, Y_t, Z_t, \Gamma_t)dW_t$$
$$dY_t^{Z,\Gamma} = Z_t \cdot dX_t + \frac{1}{2}\Gamma_t : d\langle X\rangle_t - H_t(X, Y_t^{Z,\Gamma}, Z_t, \Gamma_t)dt$$

On the function $\underline{V}$

$\underline{V}$ is a standard stochastic control problem

$\underline{V}$ can be analyzed by classical techniques from optimal control theory. In particular :

- $\underline{V}$ characterized by standard HJB equation
- Many numerical methods available for approximation...

Main result

Theorem

We have

$$V_0^P = \sup_{Y_0 \geq R} \underline{V}_0(X_0, Y_0)$$

Given maximizer $Y_0^\star$, the corresponding optimal controls $(Z^\star, \Gamma^\star)$ induce an optimal contract $\xi^\star := Y_T^{Z^\star,\Gamma^\star}$

How to prove the remaining inequality

Sufficient condition 1 : for any $\xi \in \Xi_R$, find $Y_0 \in \mathbb{R}$, $Z, \Gamma \in \mathbb{F}^X$ s.t.

$$\xi = Y_T^{Z,\Gamma} = Y_0 + \int_0^T Z_s \cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds$$

$\mathbb{P}-$ a.s. for all $\mathbb{P}$ solution of controlled SDE

where $H(t, x, y, z, \gamma)$ is the Hamiltonian of the Agent problem

Sufficient condition 2 : for any $\xi \in \Xi_R$, find a sequence $(\xi^n)_n$ s.t.

$$\xi^n = Y_T^{Z^n,\Gamma^n} \quad \text{and} \quad \xi^n \longrightarrow \xi \quad \text{in appropriate sense}$$

Outline

Reduction of the representation problem

Suppose $\sigma_t(\omega, b) = \sigma_t(\omega)$. Then Hamiltonian reduces to

$$H_t(\omega, y, z, \gamma) = F_t(\omega, y, z) + \frac{1}{2}\mathrm{Tr}[\sigma_t\sigma_t^\top(\omega)\gamma],$$

with

$$F_t(\omega, z) := \sup_u \left\{ -c_t(\omega, u) - k_t(\omega, u)y + \sigma_t(\omega)\lambda_t(\omega, a)\cdot z \right\}$$

Representation problem reduces to

$$\begin{aligned}
\xi = Y_T^{Z,\Gamma} &= Y_0 + \int_0^T Z_s\cdot dX_s + \frac{1}{2}\Gamma_s : d\langle X\rangle_s - H_s(X, Y_s^{Z,\Gamma}, Z_s, \Gamma_s)ds \\
&= Y_0 + \int_0^T Z_s\cdot dX_s - F_s(X, Y_s^{Z,\Gamma}, Z_s)ds \\
&= Y_T^{Z,0} =: Y_T^Z
\end{aligned}$$

Reduction of the representation problem

Representation problem is now

$$\xi = Y_T^Z \quad = \quad Y_0 + \int_0^T Z_s \cdot dX_s - F_s(X, Y_s^{Z,\Gamma}, Z_s)ds$$

must hold

$\mathbb{P}$ − a.s. for all $\mathbb{P}$ solution of controlled SDE

or, equivalently,

$\mathbb{P}_0$ − a.s. for some $\mathbb{P}$ solution of controlled SDE

Semilinear path-dependent HJB equation and backward SDE

Pardoux & Peng '91 : F for Lipschitz, $\xi \in \mathbb{L}^p$, $p > 1$:

$$\exists\, \mathbb{F}-\text{solution } (Y, Z) = (Y, \partial_\omega Y) \text{ with } \|Y\|_{\mathbb{L}^2} + \|\partial_\omega Y\|_{\mathbb{L}^2} < \infty$$

$\Longrightarrow$ Sobolev solution of the path-dependent semilinear PDE

$\equiv$ Representation required in SC1 for Principal-Agent problem

The case of controlled diffusion : difficulties

• Similar to the Markov case, very difficult to access to the Hessian component... Need a relaxation of the C^2-regularity

• The $\mathbb{P}^\nu$'s (measures induces by the controlled state) are defined on different supports for different values of ν, so can not reduce the analysis to one single measure

$\Longrightarrow$ Quasi-sure stochastic analysis : stochastic analysis under a non-dominated family of singular measure

From fully nonlinear HJB equation to semilinear

- $H_t(\omega, z, \gamma)$ non-decreasing and convex in γ, Then

$$H_t(\omega, z, \gamma) \;=\; \sup_{a \geq 0} \left\{ \frac{1}{2} a : \gamma - H_t^*(\omega, z, a) \right\}$$

Path-dependent HJB equation is

$$\partial_t v + \sup_{a \geq 0} \left\{ \frac{1}{2} a : \partial^2_{\omega\omega} v - H_t^*(\partial_\omega v, a) \right\} \;=\; 0, \quad v_T = \xi$$

$\Longrightarrow$ stochastic representation

$$v_t(\omega) \;=\; \sup_a \; Y_t^a(\omega)$$

where, denoting $\mathbb{P}^a := \mathbb{P}_0 \circ \left(\int_0^\cdot a_s^{1/2} dX_s \right)$,

$$Y_t^a = \xi - \int_t^T H_s^*(Z_s^a, a_s) ds + \int_t^T Z_s^a dX_s, \quad \mathbb{P}^a - \text{a.s.}$$

Wellposedness of second order BSDEs

The following statement does not refer to the appropriate integrability conditions...

Theorem (Soner, NT & Zhang '10, Possamaï, Tan & Zhou '15)

For all ξ, there exists a unique triple of $\mathbb{F}-$adapted processes (Y, Z, K), such that

- $Y_t = \xi - \int_t^T H_s^*(Z_s, a_s)ds - \int_t^T Z_s dX_s + \int_t^T dK_s$, $\mathbb{P}$a.s. *for all* $\mathbb{P}$
- K *nondecreasing,* $K_0 = 0$, *and* $\inf_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\big[K_T\big] = 0$

This is our

Sobolev solution of the fully-nonlinear path-dependent PDE...

Regularity reduces to the non-decreasing process K

Suppose $K_t = \int_0^t \dot{K}_s ds,\ t \in [0, T]$, and define the process Γ by

$$\dot{K}_t \quad = \quad H_t(Z_t, \Gamma_t) - \frac{1}{2} a_t : \Gamma_t + H_t^*(Z_t, a_t)$$

Substituting in the 2BSDE, we get for all a :

$$\begin{aligned} Y_t &= \xi - \int_t^T H_s^*(Z_s, a_s) ds - \int_t^T Z_s dX_s + \int_t^T \dot{K}_s ds \\ &= \xi + \int_t^T H_s(Z_s, \Gamma_s) ds - \frac{1}{2} \Gamma_s : d\langle X \rangle_s - Z_s dX_s,\ \mathbb{P}^a - \text{a.s.} \end{aligned}$$

$\equiv$ Representation required in SC1 for Principal-Agent problem

$\Longrightarrow$ $Y_t(\omega)$ solves the path-dependent HJB equation :

$$\partial_t Y + H_t(\partial_\omega Y, \partial^2_{\omega\omega} Y) = 0, \quad Y_T = \xi$$

In the fully nonlinear case, the problem is solved by verifying SC2 (approximating K by a.c. processes)

THANK YOU FOR YOUR ATTENTION