# Calibration of Dupire's Local Volatility Models from Option Data

Jorge P. Zubelli

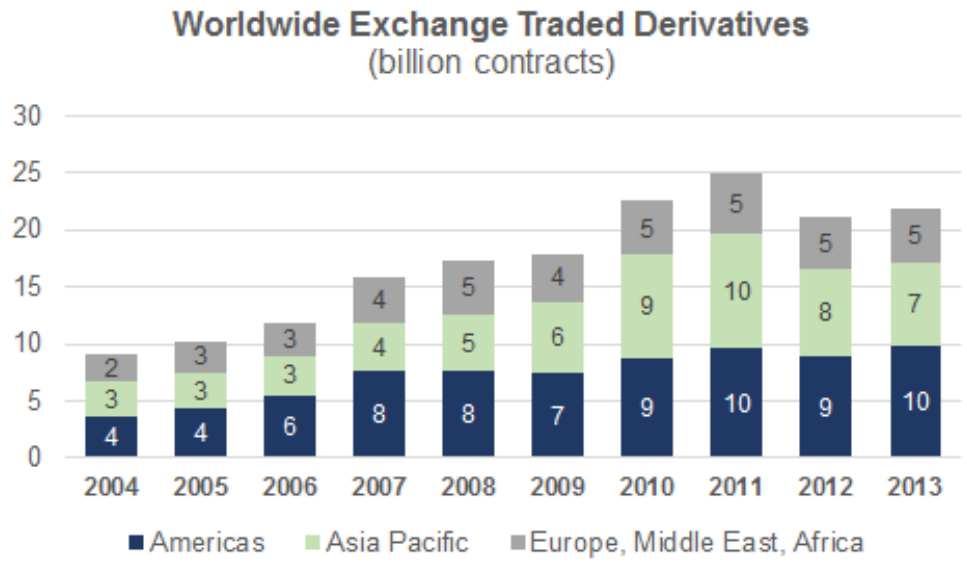
IMPA

Thanks to the
Organizing Committee
in particular to C. Sagastizabal

Mar. 28th, 2016

# Outline

# Derivative Markets



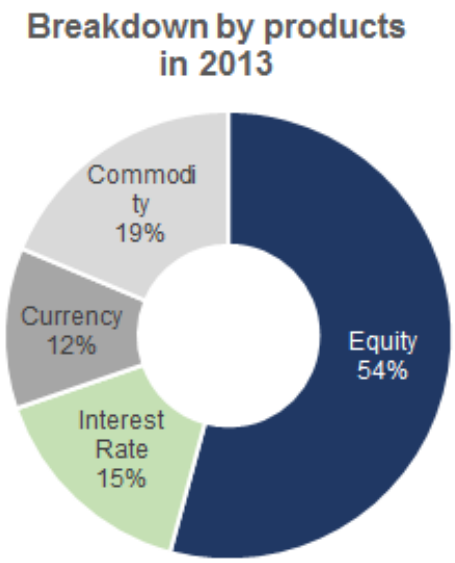




Figure: In 2013, commodities represented 19% of the total amount of traded derivatives. Source: World Federation of Exchanges

# Black-Scholes Market Model

Assume two assets: a risky stock and a riskless bond.

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

$$\mathrm{d}\beta_t = r\beta_t \mathrm{d}t.$$

Price of an option at time $P(t,x)$ at time $t$ and spot value $x$:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + (r-\delta)x\frac{\partial P}{\partial x} - rP = 0 \qquad P(T_E,\cdot) = h \tag{1}$$

where $h$ is the payoff at time $T_E$ and $\delta$ is the continuous dividend rate.

**Note:** In the original model $\sigma$ is constant.

# However...

- Volatility is not constant! not even deterministic! It a multi-scale phenomena!
- It is not true that the underlying undergoes an Exponential Brownian Motion
- Even more so in high frequency contexts...

# However...

- Volatility is not constant! not even deterministic! It a multi-scale phenomena!
- It is not true that the underlying undergoes an Exponential Brownian Motion
- Even more so in high frequency contexts...

**Implied Volatility:** The value of the volatility that should be used in the Black-Scholes formula to give the quoted market price of a derivative.

# The Concept of Implied Volatility

Recall

$$C^{\mathrm{BS}}(X,t;K,T,r,\sigma_0) = XN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{2}$$

where $N$ is the cumulative normal distribution function and

$$d_\pm = \frac{\log(Xe^{r(T-t)}/K)}{\sigma_0\sqrt{T-t}} \pm \frac{\sigma_0\sqrt{T-t}}{2}\,. \tag{3}$$

# The Concept of Implied Volatility

Recall

$$C^{\mathrm{BS}}(X,t;K,T,r,\sigma_0) = XN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{2}$$

where $N$ is the cumulative normal distribution function and

$$d_\pm = \frac{\log(Xe^{r(T-t)}/K)}{\sigma_0\sqrt{T-t}} \pm \frac{\sigma_0\sqrt{T-t}}{2}\,. \tag{3}$$

**Notion of Implied Volatility:** Fix everything else and consider

$$\sigma \longmapsto C^{\mathrm{BS}}(X,t;K,T,r,\sigma)$$

The implied volatilty is the inverse to this map.

# The Concept of Implied Volatility

Recall

$$C^{\mathrm{BS}}(X,t;K,T,r,\sigma_0) = XN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{2}$$

where $N$ is the cumulative normal distribution function and

$$d_\pm = \frac{\log(Xe^{r(T-t)}/K)}{\sigma_0\sqrt{T-t}} \pm \frac{\sigma_0\sqrt{T-t}}{2}\,. \tag{3}$$

**Notion of Implied Volatility:** Fix everything else and consider

$$\sigma \longmapsto C^{\mathrm{BS}}(X,t;K,T,r,\sigma)$$

The implied volatilty is the inverse to this map.

**IMPLIED VOL:** ”wrong number that when plugged into the wrong equation gives the right price”

# IMPLIED VOL

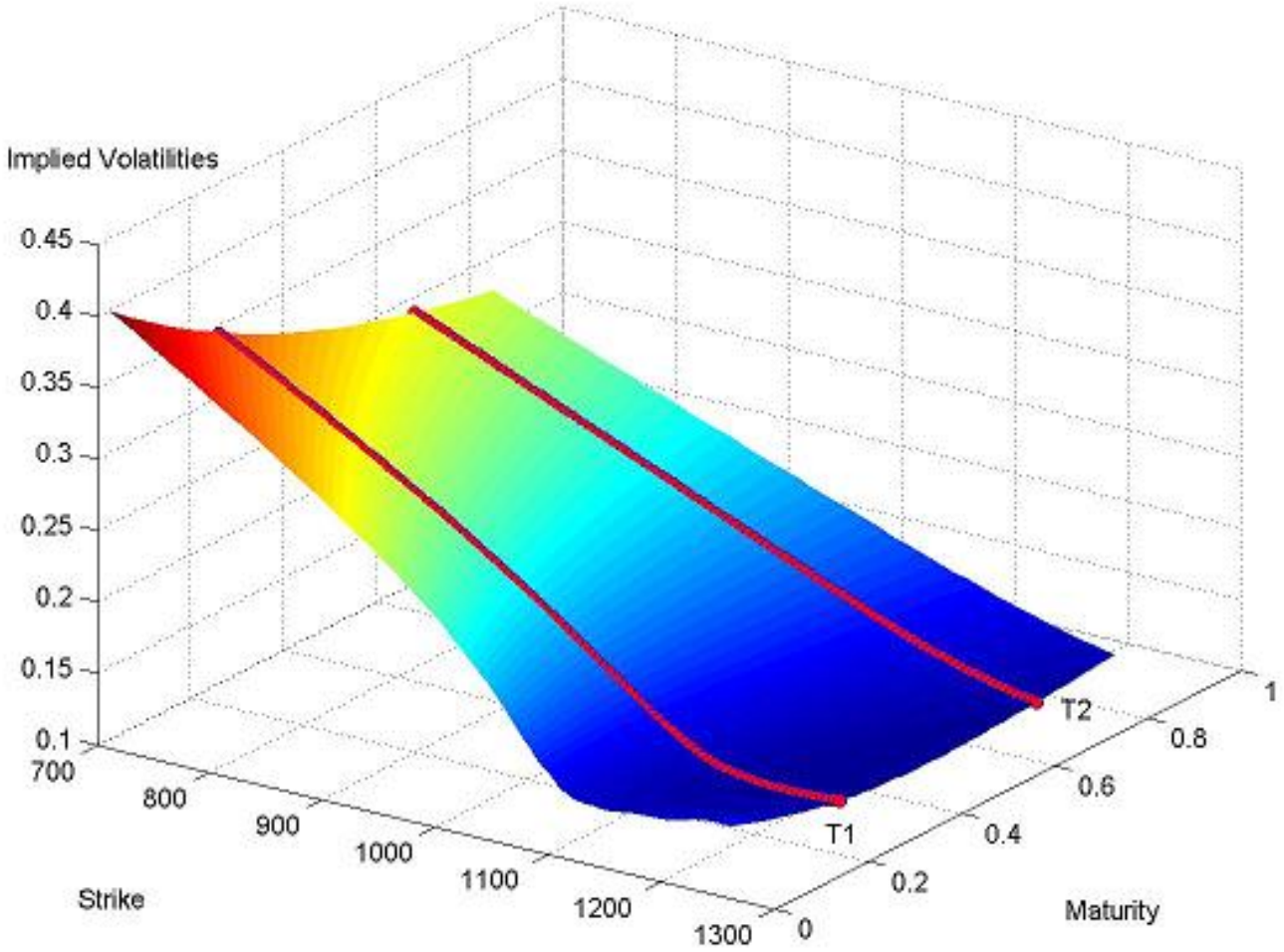
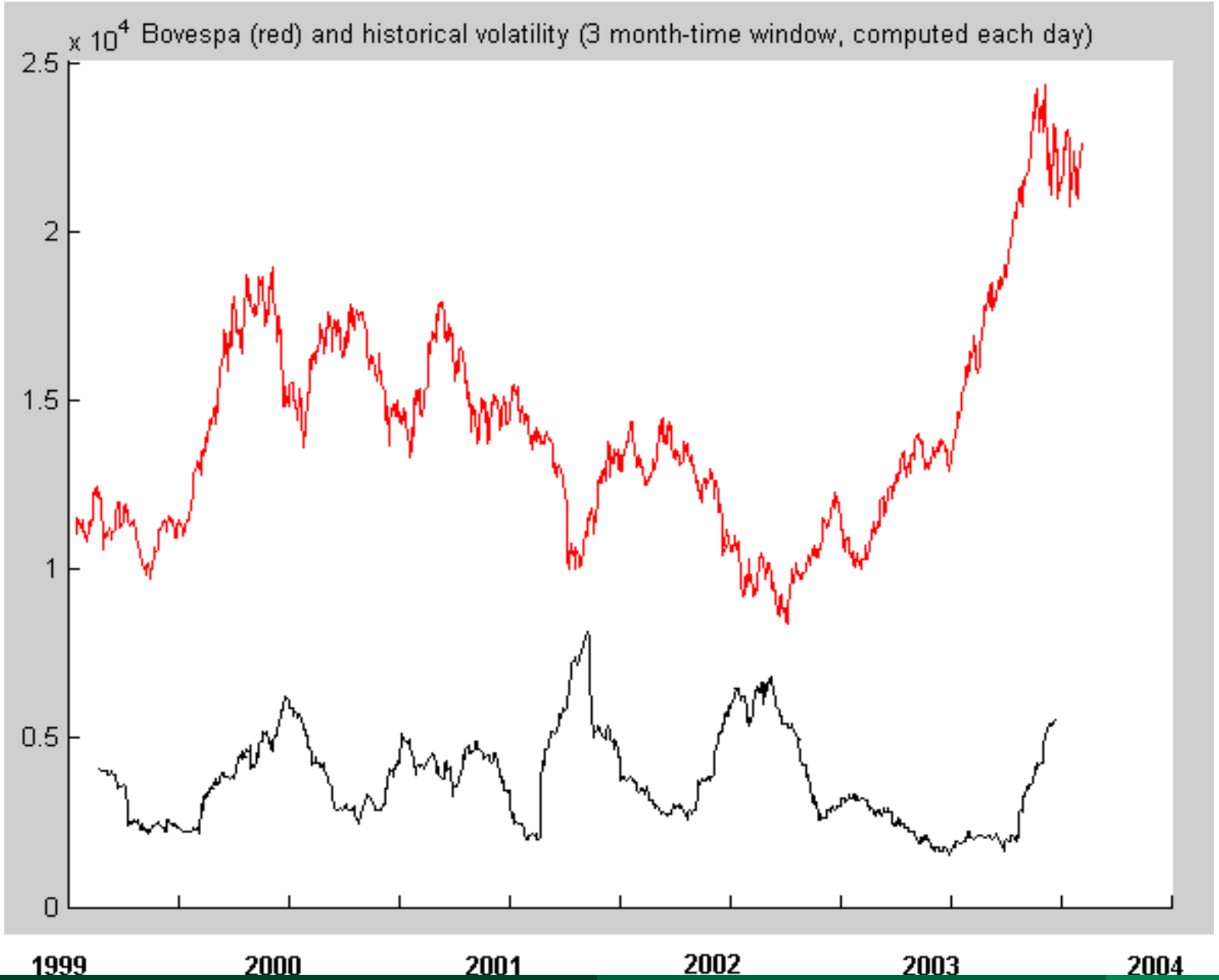



Figure: Implied Volatility Surface- (From Bruno Dupire - IMPA talk)

# Stochastic Behavior of the Volatility

## IBOVESPA Index and its Volatility



# Model Uncertainty

The concept of **Model Risk** is as important as operational, market, and credit risk

# Model Uncertainty

The concept of **Model Risk** is as important as operational, market, and credit risk

**Report Williams(1999):** $5 billion in derivative losses during 1999 were attributable to model risk.

# Model Uncertainty

The concept of **Model Risk** is as important as operational, market, and credit risk

**Report Williams(1999):** $5 billion in derivative losses during 1999 were attributable to model risk.

*Fitting the smile:* Local volatity reconstruction/identification...

# Model Uncertainty

The concept of **Model Risk** is as important as operational, market, and credit risk

**Report Williams(1999):** $5 billion in derivative losses during 1999 were attributable to model risk.

*Fitting the smile:* Local volatity reconstruction/identification...

We'll see that this is not a well-posed problem in Hadamard's sense

# Model Uncertainty

The concept of **Model Risk** is as important as operational, market, and credit risk

**Report Williams(1999):** $5 billion in derivative losses during 1999 were attributable to model risk.

*Fitting the smile:* Local volatity reconstruction/identification...

We'll see that this is not a well-posed problem in Hadamard's sense It needs stabilization!

# Volatility

## Different Interpretations

- Econometrics - Historical
- Implied (or Implicit)
- Stochastic Volatility Models
  - fast mean reversion (Papanicolaou, Fouque, et al)
  - for commodities: **jt work Fouque, Saporito, Zubelli; IJTAF2015**
- Local Volatility NON PARAMETRIC (focus of this talk)

# Central Problems

- Understand volatility behavior.

# Central Problems

- Understand volatility behavior.
- Protect portfolios against volatility oscilations.

# Central Problems

- Understand volatility behavior.
- Protect portfolios against volatility oscilations.
- Find parsimonious and efficient models

# Central Problems

- Understand volatility behavior.
- Protect portfolios against volatility oscilations.
- Find parsimonious and efficient models (simple but not too simple!)
- Calibrate such models in a robust and effective way.

# Central Problems

- Understand volatility behavior.
- Protect portfolios against volatility oscilations.
- Find parsimonious and efficient models (simple but not too simple!)
- Calibrate such models in a robust and effective way.
- Price other derivatives consistently

# Local Volatility Model

B. Dupire

**Idea:** Assume that the volatility is given by

$$\sigma = \sigma(t, X)$$

i.e.: it depends on time and the asset price.

# Local Volatility Model

B. Dupire

**Idea:** Assume that the volatility is given by

$$\sigma = \sigma(t, X)$$

i.e.: it depends on time and the asset price.
Easy to check that the Black-Scholes eq. holds.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma(t,X)^2 X^2 \frac{\partial^2 P}{\partial X^2} + r\left(X\frac{\partial P}{\partial X} - P\right) = 0 \tag{4}$$

$$P(T, X) = h(X) \tag{5}$$

# Local Volatility Model

B. Dupire

**Idea:** Assume that the volatility is given by

$$\sigma = \sigma(t, X)$$

i.e.: it depends on time and the asset price.
Easy to check that the Black-Scholes eq. holds.

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma(t,X)^2 X^2 \frac{\partial^2 P}{\partial X^2} + r\left(X\frac{\partial P}{\partial X} - P\right) = 0 \tag{4}$$

$$P(T, X) = h(X) \tag{5}$$

**From now on:** $h(X) = (X - K)^+$ or $h(X) = (K - X)^+$

# The Smile Curve and Dupire's Equation

Assuming that there exists a local volatility function $\sigma = \sigma(t, X)$ for which (4) holds Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T C - \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 C + rK\partial_K C = 0\ , \quad K > 0\ ,\ T \geq 0 \\ C(K, T = 0) = (X - K)^+\ , \end{cases} \tag{6}$$

**Theoretical:** way of evaluating the local volatility

# The Smile Curve and Dupire's Equation

Assuming that there exists a local volatility function $\sigma = \sigma(t, X)$ for which (4) holds Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T C - \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 C + rK\partial_K C = 0\ , \quad K > 0\ ,\ T \geq 0 \\ C(K, T = 0) = (X - K)^+\ , \end{cases} \tag{6}$$

**Theoretical:** way of evaluating the local volatility

$$\sigma(T,K) = \sqrt{2\left(\frac{\partial_T C + rK\partial_K C}{K^2\partial_K^2 C}\right)} \tag{7}$$

*In practice* To estimate $\sigma$ from (6), limited amount of discrete data and thus interpolate.

# The Smile Curve and Dupire's Equation

Assuming that there exists a local volatility function $\sigma = \sigma(t, X)$ for which (4) holds Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T C - \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 C + rK\partial_K C = 0\,, \quad K > 0\,,\, T \geq 0 \\ C(K, T = 0) = (X - K)^+\,, \end{cases} \tag{6}$$

**Theoretical:** way of evaluating the local volatility

$$\sigma(T,K) = \sqrt{2\left(\frac{\partial_T C + rK\partial_K C}{K^2\partial_K^2 C}\right)} \tag{7}$$

*In practice* To estimate $\sigma$ from (6), limited amount of discrete data and thus interpolate. Numerical instabilities! Even to keep the argument positive is hard.

# Related Work

Very vast!!!

- Avellaneda et al. [ABF$^+$00, Ave98c, Ave98b, Ave98a, AFHS97]
- Bouchev & Isakov [BI97]
- Crepey [Cré03]
- Derman et al. [DKZ96]
- Egger & Engl [EE05]
- Hofmann et al. [HKPS07, HK05]
- Jermakyan [BJ99]
- Achdou & Pironneau (2004)
- Roger Lee (2001,2005)
- Abken et al. (1996)
- Ait Sahalia, Y & Lo, A (1998)
- Berestycki et al. (2000)
- Buchen & Kelly (1996)
- Coleman et al. (1999)
- Cont, Cont & Da Fonseca (2001)
- Jackson et al. (1999)
- Jackwerth & Rubinstein (1998)
- Jourdain & Nguyen (2001)
- Lagnado & Osher (1997)
- Samperi (2001)
- Stutzer (1997)

# Message

- Local Vol. calibration is an important problem
- Ill-posed problem that requires regularization
  - Lots of numerical issues
  - Convex optimization tools
  - Data assimilation problems, noisy, bid & ask spreads, model noisy
  - Bayesian interpretation
- Present techniques applicable also to commodities
- Current research: Heston with a local-vol term...
- Future research: Integrate with exotic option pricing

# Impact

- Pricing of exotic options
- Risk management
- Volatility trading
- Uncertainty quantification and model risk reduction

# Problem Statement

Starting Point: Dupire forward equation [Dup94]

$$-\partial_T U + \frac{1}{2}\sigma^2(T,K)K^2\partial_K^2 U - (r-q)K\partial_K U - qU = 0\,, \quad T > 0\,, \tag{8}$$

$$K = X_0 e^y\,,\ \tau = T - t\,,\ b = q - r\,, \quad u(\tau, y) = e^{q\tau} U^{t,X}(T,K) \tag{9}$$

and

$$a(\tau, y) = \frac{1}{2}\sigma^2(T - \tau; X_0 e^y)\,, \tag{10}$$

Set $q = r = 0$ for simplicity to get:

$$u_\tau = a(\tau, y)(\partial_y^2 u - \partial_y u) \tag{11}$$

and initial condition

$$u(0, y) = X_0(1 - e^y)^+ \tag{12}$$

# Problem Statement

## The Vol Calibration Problem

Given an observed set

$$\{u = u(t, X, T, K; \sigma)\}_{(T,K)\in\mathcal{X}}$$

find $\sigma = \sigma(t, X)$ that best fits such market data

**Noisy data:** $u = u^{\delta}$

Admissible *convex* class of calibration parameters:

$$\mathcal{D}(F) := \{a \in a_0 + H^{1+\varepsilon}(\Omega) : \underline{a} \leq a \leq \overline{a}\}. \tag{13}$$

where, for $0 \leq \varepsilon$ fixed, $U := H^{1+\varepsilon}(\Omega)$ and $\overline{a} > \underline{a} > 0$.

## Parameter-to-solution operator

$F : \mathcal{D}(F) \subset H^{1+\varepsilon}(\Omega) \to L^2(\Omega)$

$$F(a) = u(a) \tag{14}$$

# Setting of the problem

**Theorem (H. Egger-H. Engl[EE05] Crepey[Cré03])**

*The parameter to solution map*

$$F : H^{1+\varepsilon}(\Omega) \to L^2(\Omega)$$

*is*

- *weak sequentialy continuous*
- *compact and weakly closed*

**Consequences:**

- The inverse problem is ill-posed.
- We can prove that the inverse problem satisfies the conditions to apply the regularization theory.

# Well-Posed and Ill-Posed Problems

**Hadamard's definition of well-posedness:**

- Existence
- Uniqueness
- Stability

# Well-Posed and Ill-Posed Problems

**Hadamard's definition of well-posedness:**

- Existence
- Uniqueness
- Stability

**The problem under consideration:** Ill-posed.

**Equation:**

$$F(a) = u$$

# Well-Posed and Ill-Posed Problems

**Hadamard's definition of well-posedness:**

- Existence
- Uniqueness
- Stability

**The problem under consideration:** Ill-posed.

**Equation:**

$$F(a) = u$$

**Need Regularization:**

# Approach

## Convex Tikhonov Regularization

For given convex $f$ minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{15}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

Remark that $f$ incorporates the *a priori* info on $a$.

# Approach

## Convex Tikhonov Regularization

For given convex $f$ minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{15}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

Remark that $f$ incorporates the *a priori* info on $a$.

$$||\bar{u} - u^\delta||_{L^2(\Omega)} \leq \delta, \tag{16}$$

where $\bar{u}$ is the data associated to the actual value $\hat{a} \in \mathcal{D}(F)$.

# Approach

## Convex Tikhonov Regularization

For given convex $f$ minimize the Tikhonov functional

$$\mathcal{F}_{\beta,u^\delta}(a) := ||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a) \tag{15}$$

over $\mathcal{D}(F)$, where, $\beta > 0$ is the regularization parameter.

Remark that $f$ incorporates the *a priori* info on $a$.

$$||\bar{u} - u^\delta||_{L^2(\Omega)} \leq \delta, \tag{16}$$

where $\bar{u}$ is the data associated to the actual value $\hat{a} \in \mathcal{D}(F)$.

## Assumption (very general!)

Let $\varepsilon \geq 0$ be fixed. $f : \mathcal{D}(f) \subset H^{1+\varepsilon}(\Omega) \longrightarrow [0, \infty]$ is a convex, proper, coercive and sequentially weakly lower semi-continuous functional with domain $\mathcal{D}(f)$ containing $\mathcal{D}(F)$.

# Questions

Theoretical Questions:

- Does there exist a minimizer of the regularized problem?

# Questions

Theoretical Questions:

- Does there exist a minimizer of the regularized problem?
- Suppose that the noise level goes to zero... How fast does the regularized go to the true solution?

# Questions

Theoretical Questions:

- Does there exist a minimizer of the regularized problem?
- Suppose that the noise level goes to zero... How fast does the regularized go to the true solution?

Results obtained in **joint work with D. Cezaro and O. Scherzer. Published in J. Nonlinear Analysis, 2012 [DCSZ12]**

# Practical Questions

- Can we devise an iterative algorithm to compute the solution?
- Does this algorithm converge?
- Can we regularize by stopping the iteration judiciously?

We proved:

1. **A tangential cone condition that ensures convergence of the Landwebber iteration. Joint work w/ D. Cezaro.** (IMA J. of Applied Math. 2013)
2. **Obtained a Morozov-type criterium to stop the iteration. Joint work w/ Albani & D. Cezaro** (A.Analysis & Discrete Math. 2014)
3. **Developed a regularization by discretization with a stopping criterium. Joint work w/ Albani & D. Cezaro.** (Inv. Problems in Imaging. 2016)

**We implemented:** The different algorithms and compared with alternatives (such as (ensemble) Kalman filter based iterations)

# How about algorithms?

**NOTE:** We have proved

We have also proved a tangential cone condition for this problem, which implies that the Landwever iteration converges in a suitable neighborhood. **Landweber Iteration [EHN96]:**

$$a_{k+1}^{\delta} = a_k^{\delta} + cF'(a_k^{\delta})^*(u^{\delta} - F(a_k^{\delta})). \tag{17}$$

# How about algorithms?

**NOTE:** We have proved

We have also proved a tangential cone condition for this problem, which implies that the Landwever iteration converges in a suitable neighborhood. **Landweber Iteration [EHN96]:**

$$a_{k+1}^{\delta} = a_k^{\delta} + cF'(a_k^{\delta})^*(u^{\delta} - F(a_k^{\delta})). \tag{17}$$

**Discrepancy Principle:**

$$\left\| u^{\delta} - F(a_{k_*(\delta,y^{\delta})}^{\delta}) \right\| \leq r\delta < \left\| u^{\delta} - F(a_k^{\delta}) \right\|, \tag{18}$$

where

$$r > 2\frac{1+\eta}{1-2\eta}, \tag{19}$$

is a relaxation term.

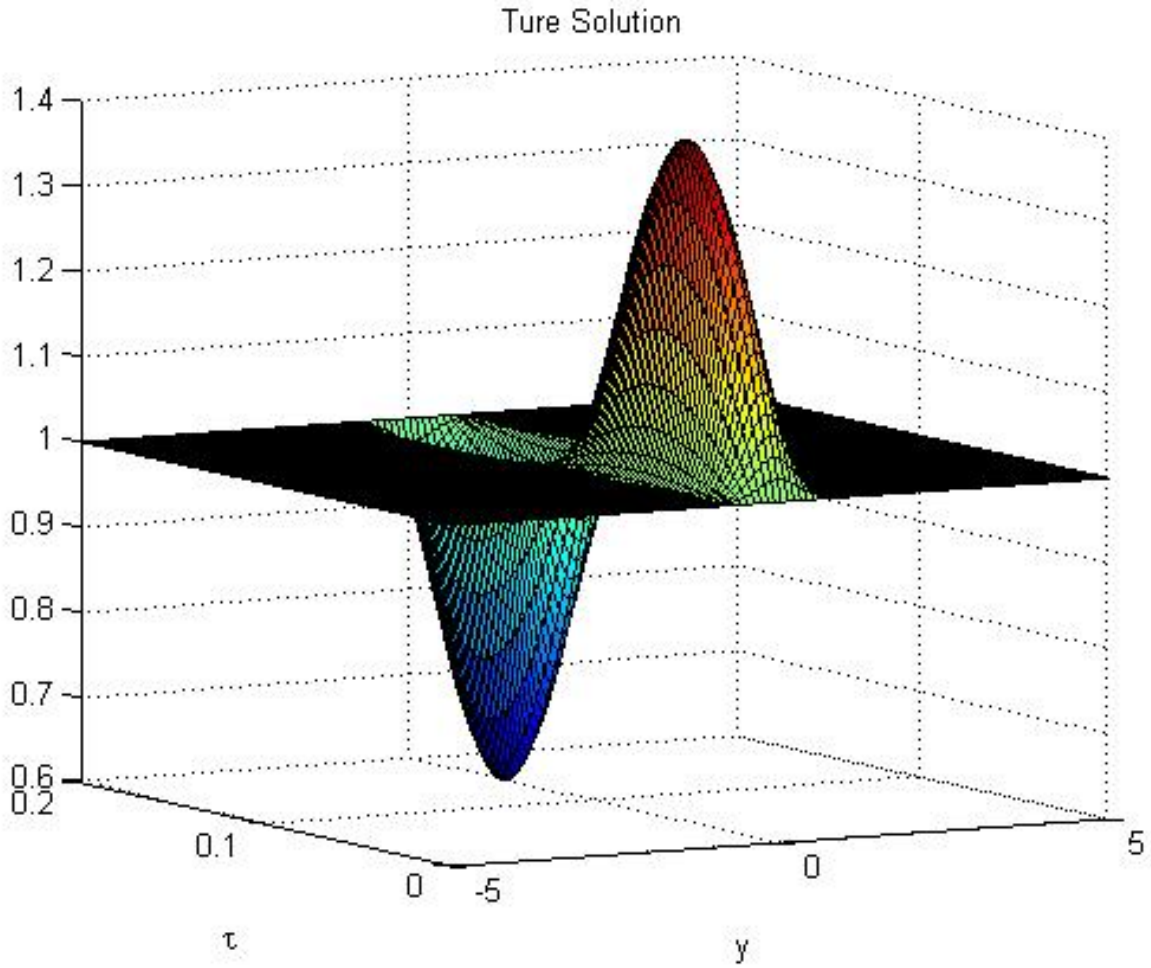
If the iteration is stopped at index $k_*(\delta, y^{\delta})$ such that for the first time, the residual becomes small compared to the quantity $r\delta$.

# Numerical Examples with Simulated Data

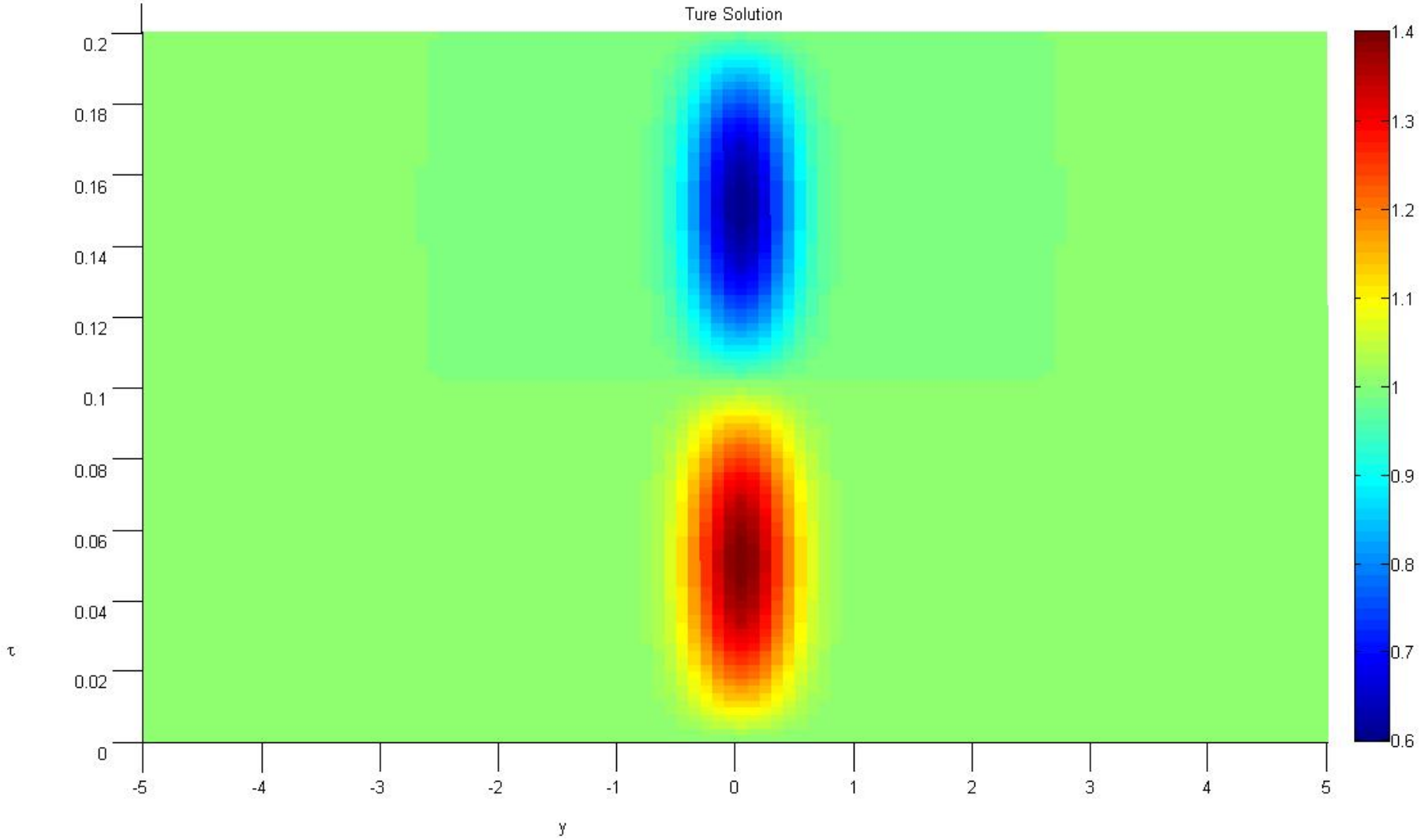
## Description of the Examples

- In our first examples we used a Landweber iteration technique we implemented the calibration.
- Produced for different test variances $a$ the option prices and added different levels of multiplicative noise.
- The examples consisted of perturbing $a = 1$ during a period of $T = 0, \cdots, 0.2$ and log-moneyness $y$ varying between $-5$ and 5.
- Initial guess: Constant volatility.

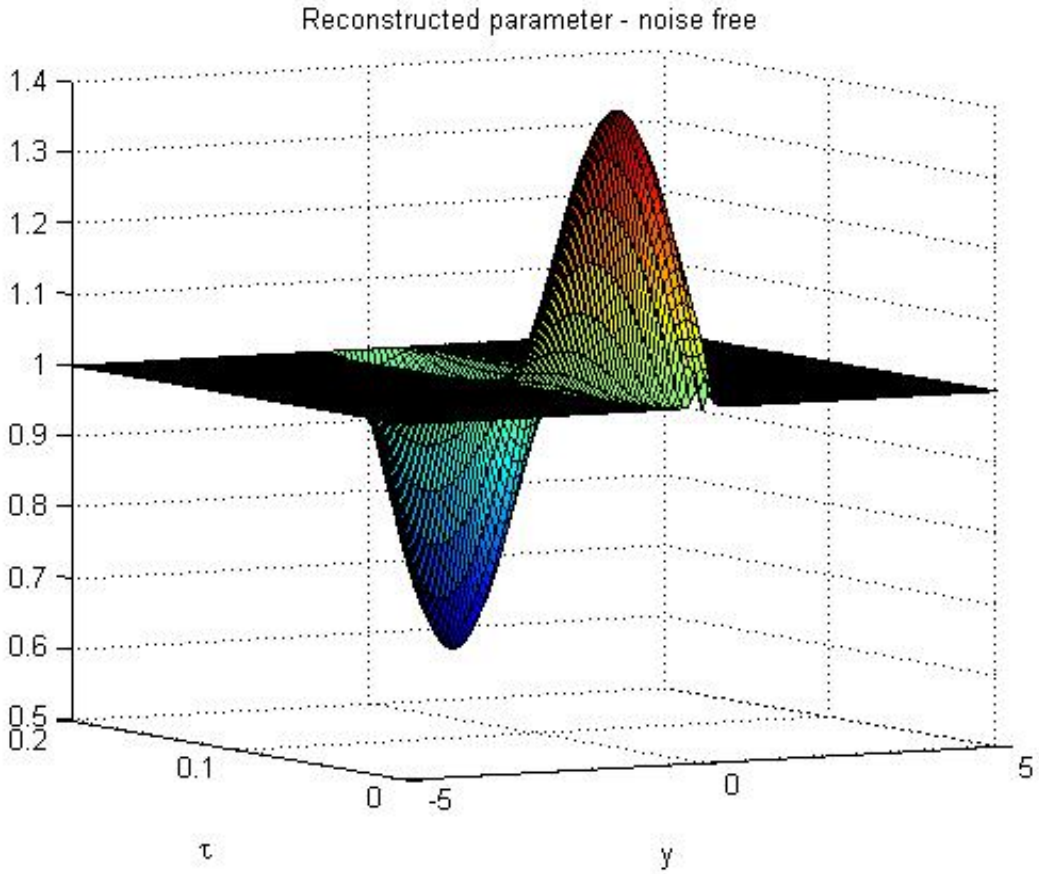
# Numerical Examples - Exact Solution



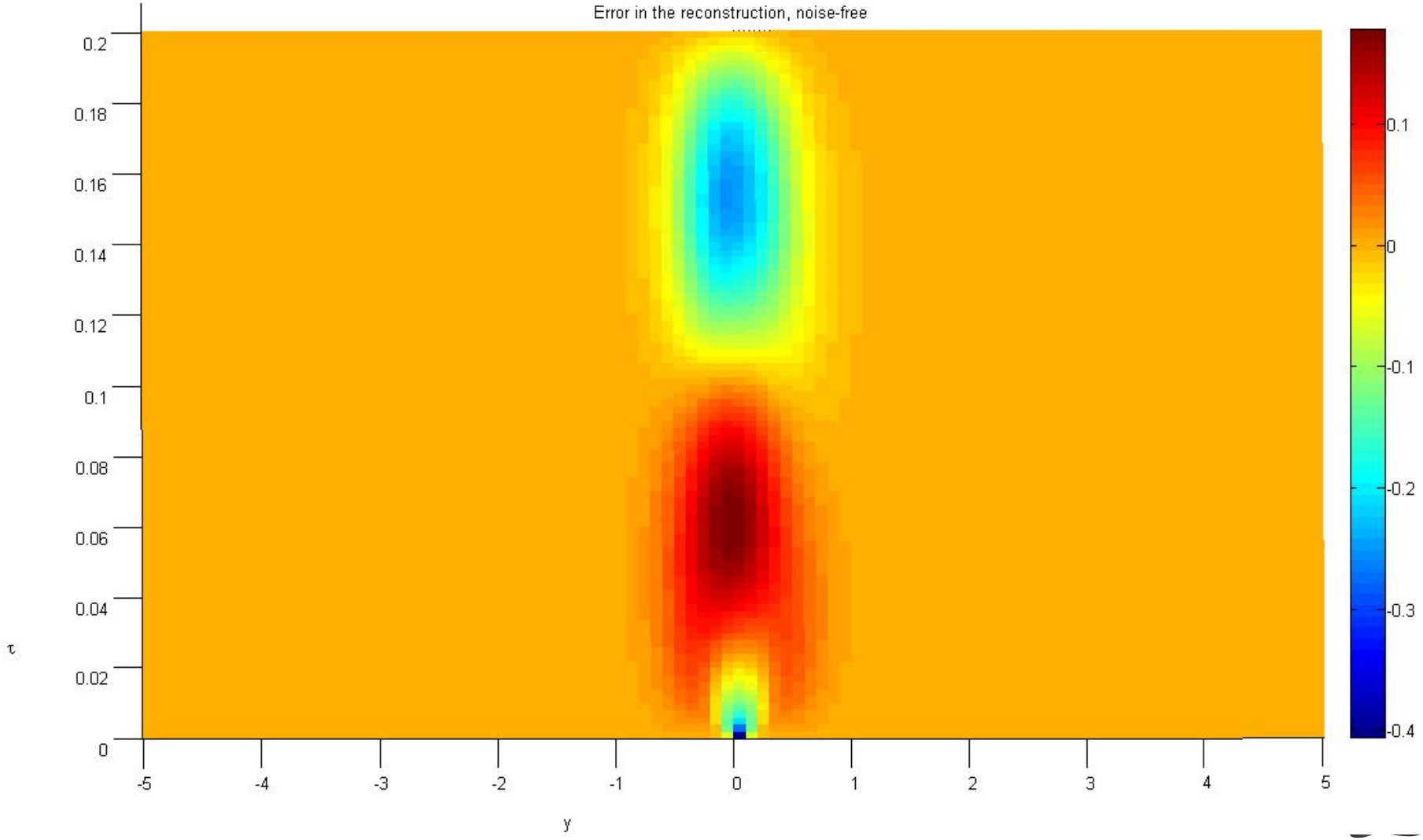
# Numerical Examples - Exact Solution



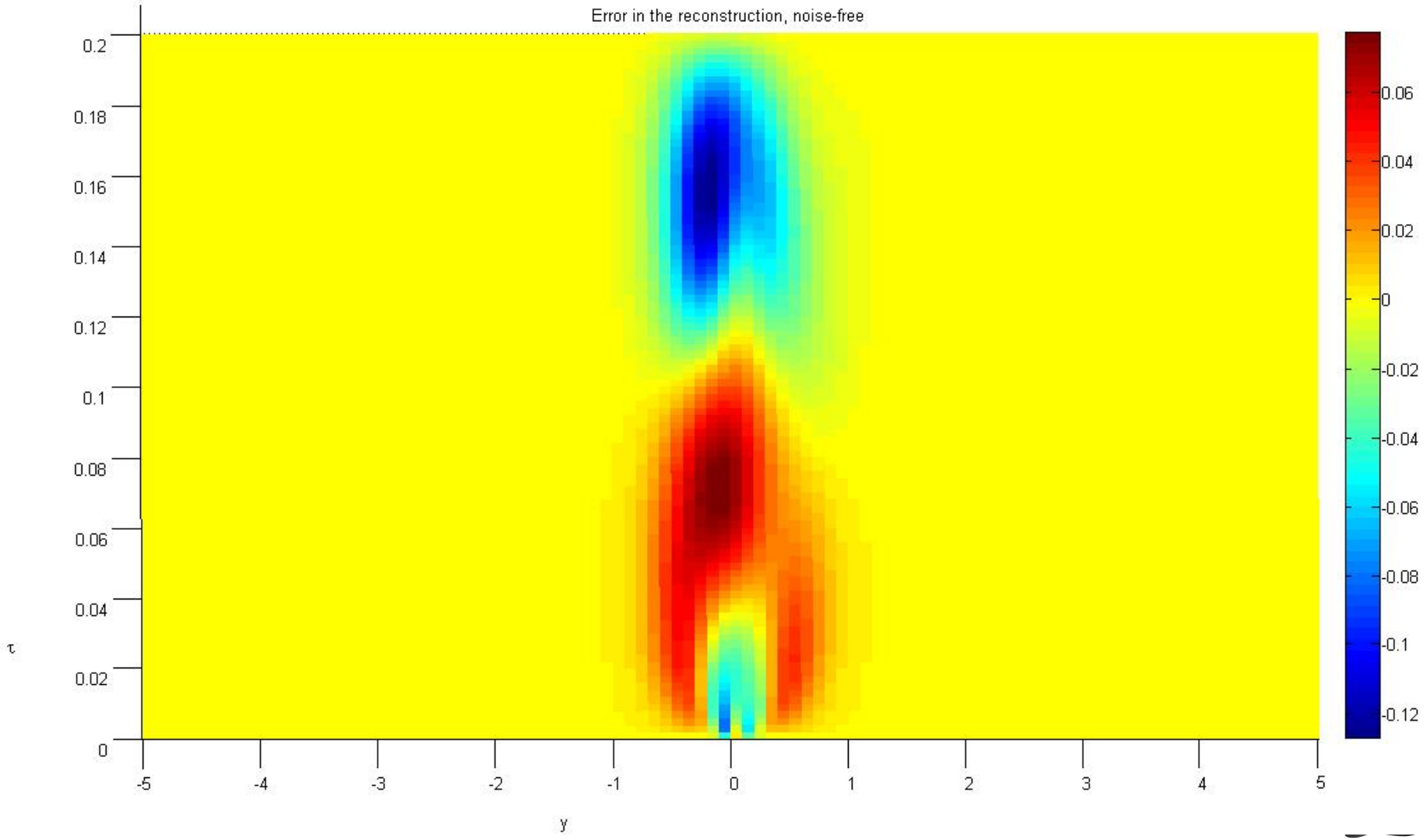
# Numerical Examples 1 - noiseless - 4000 steps



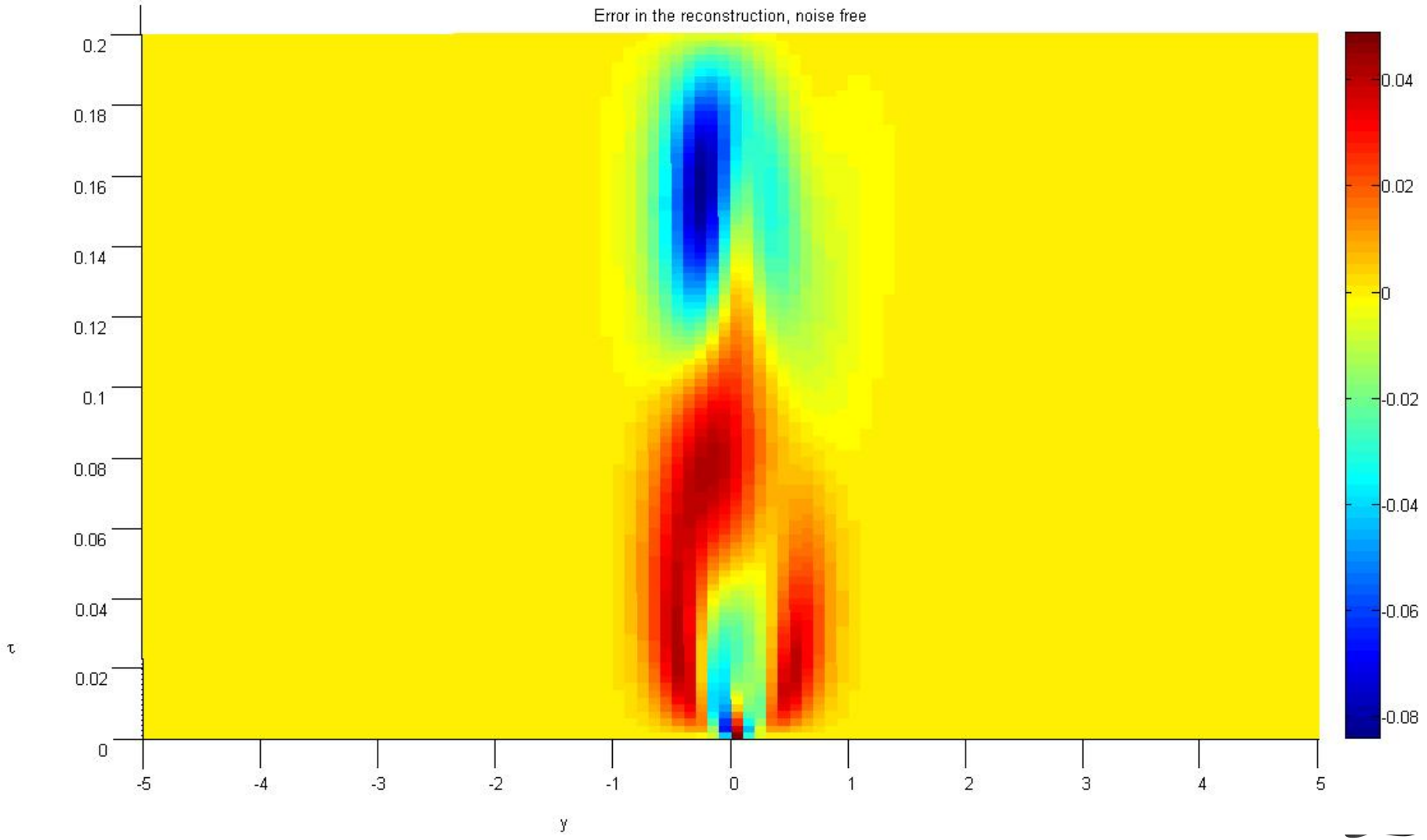
# Numerical Examples 1 - error - 100 steps



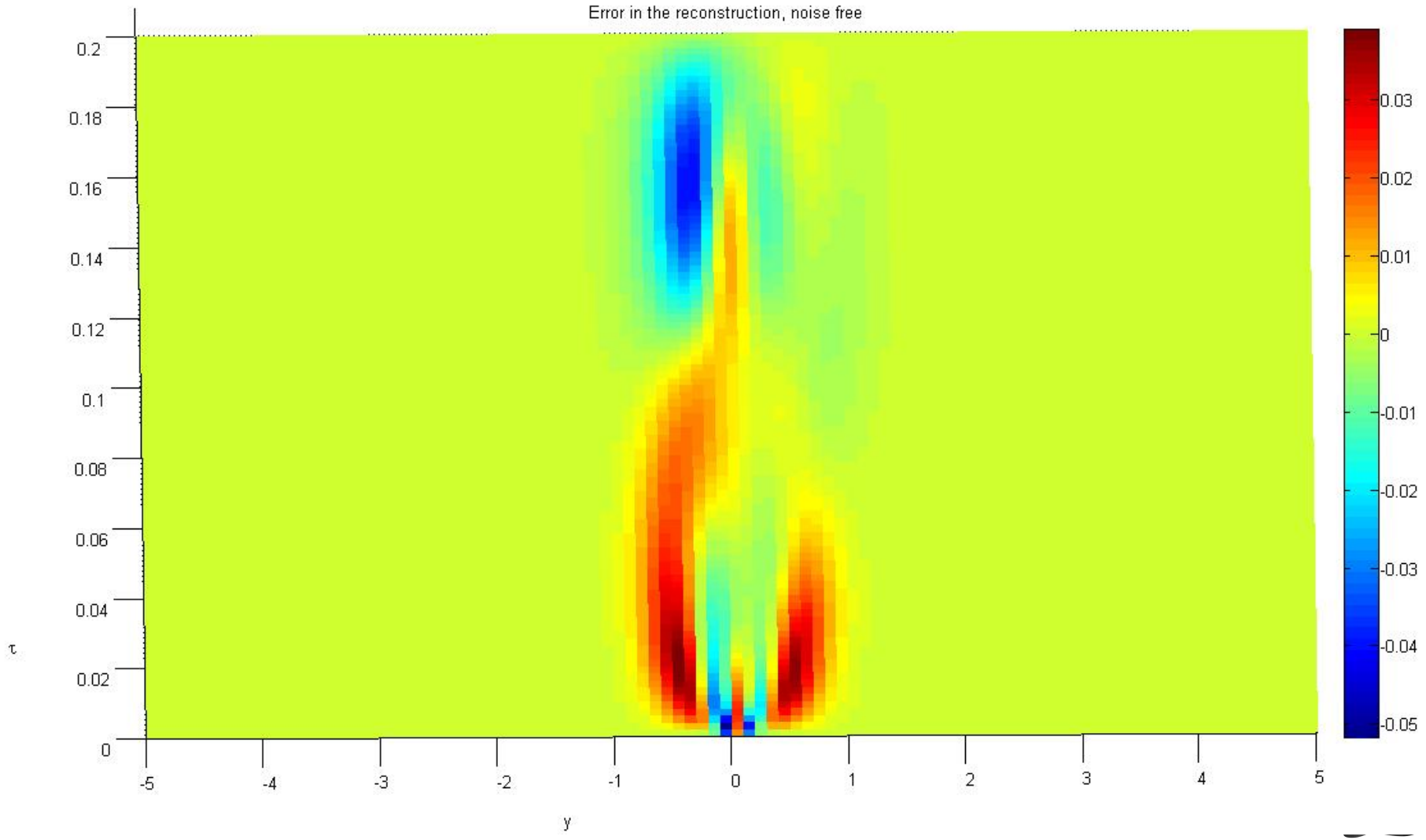
# Numerical Examples 1 - error - 300 steps



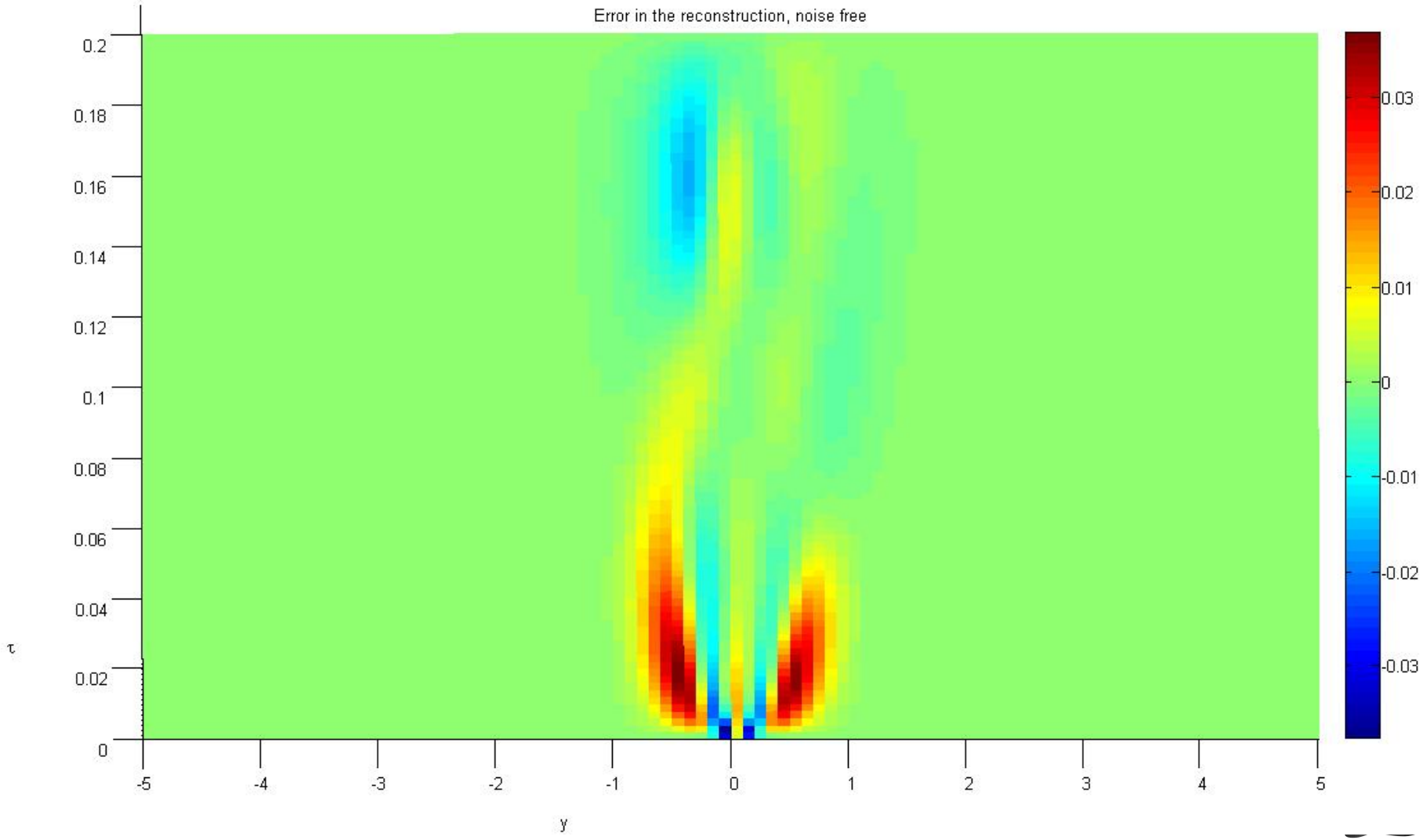
# Numerical Examples 1 - error - 500 steps



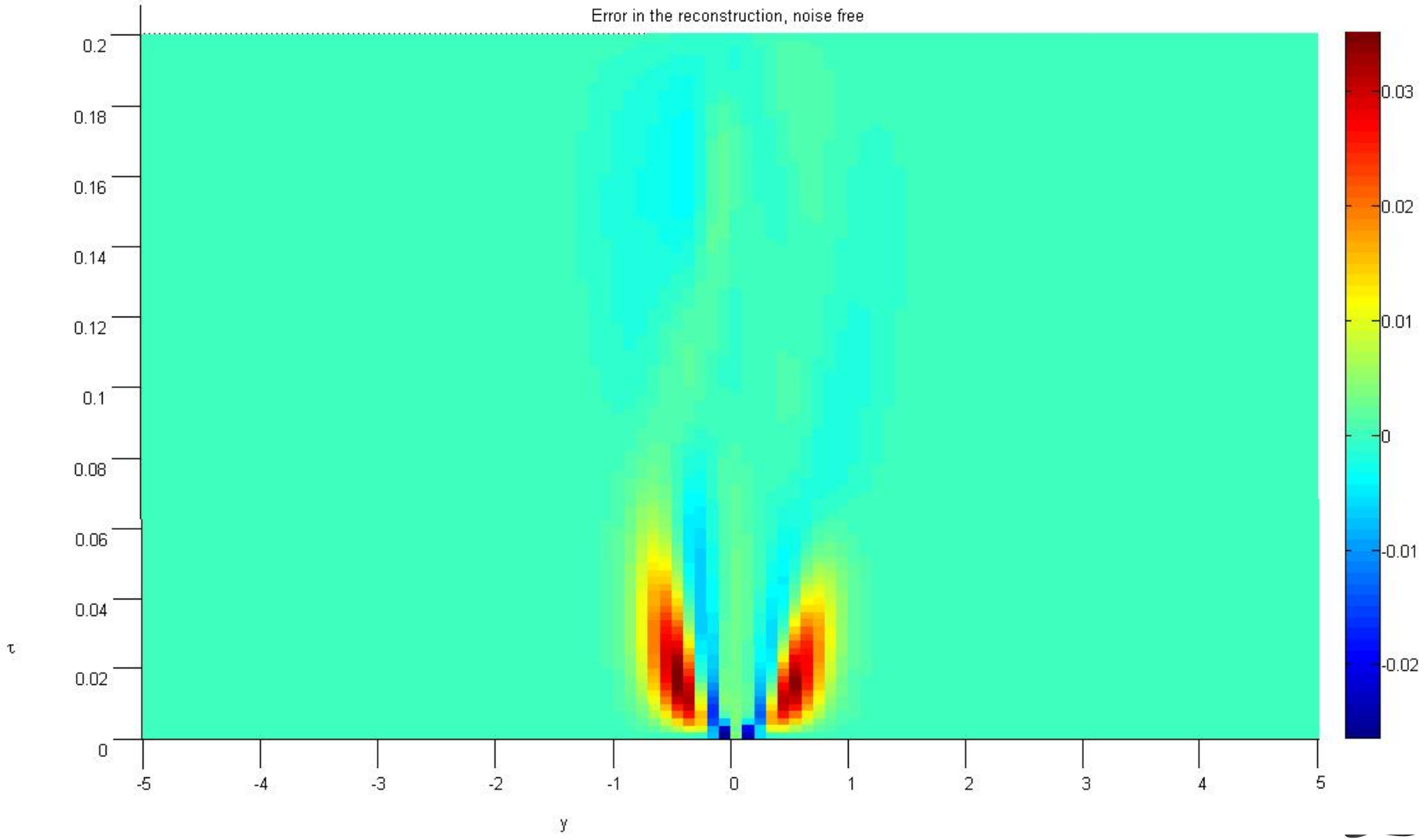
# Numerical Examples 1 - error - 1000 steps

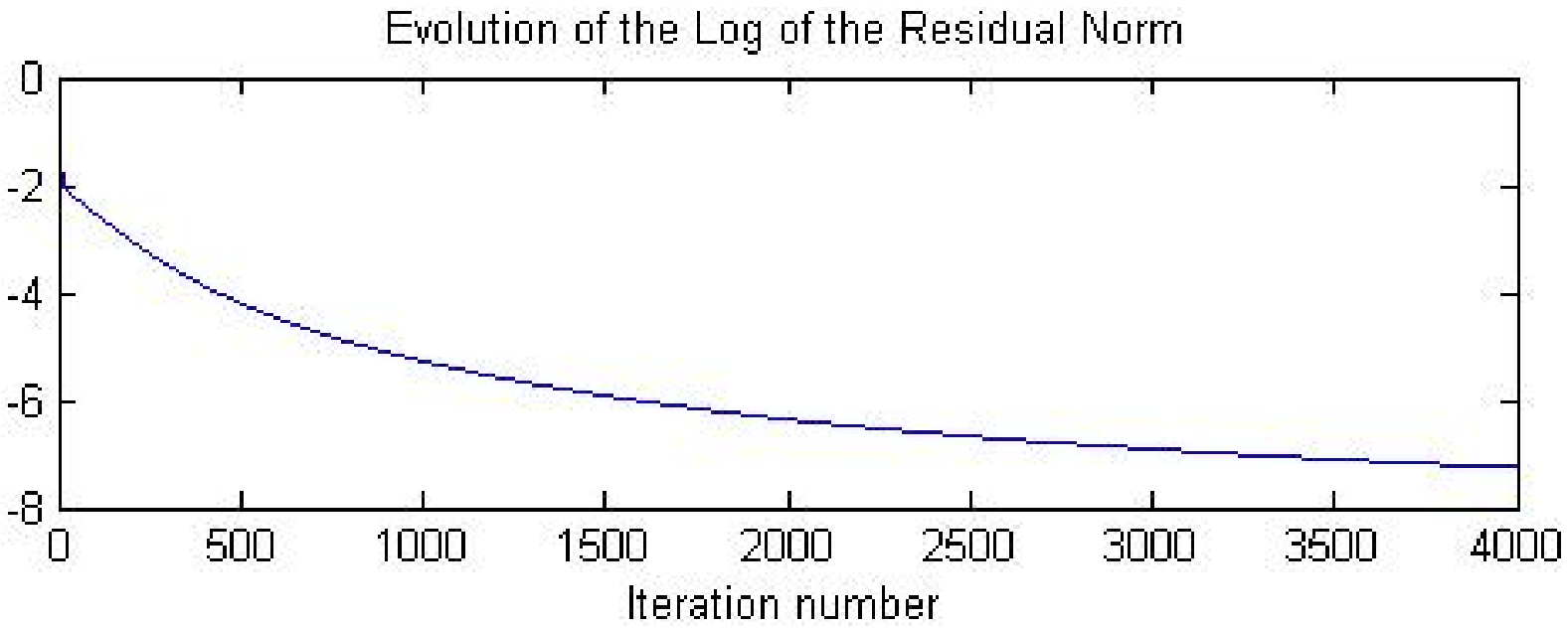


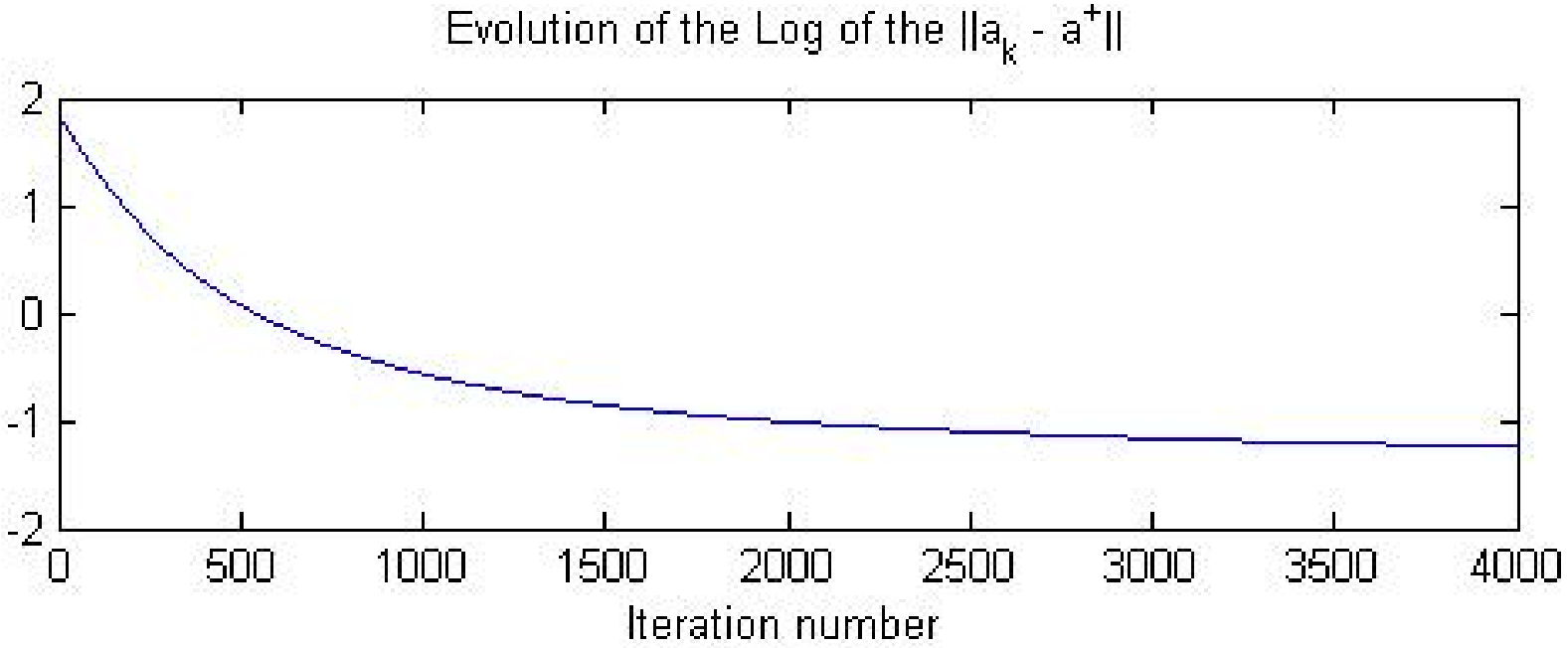
# Numerical Examples 1 - error - 2000 steps



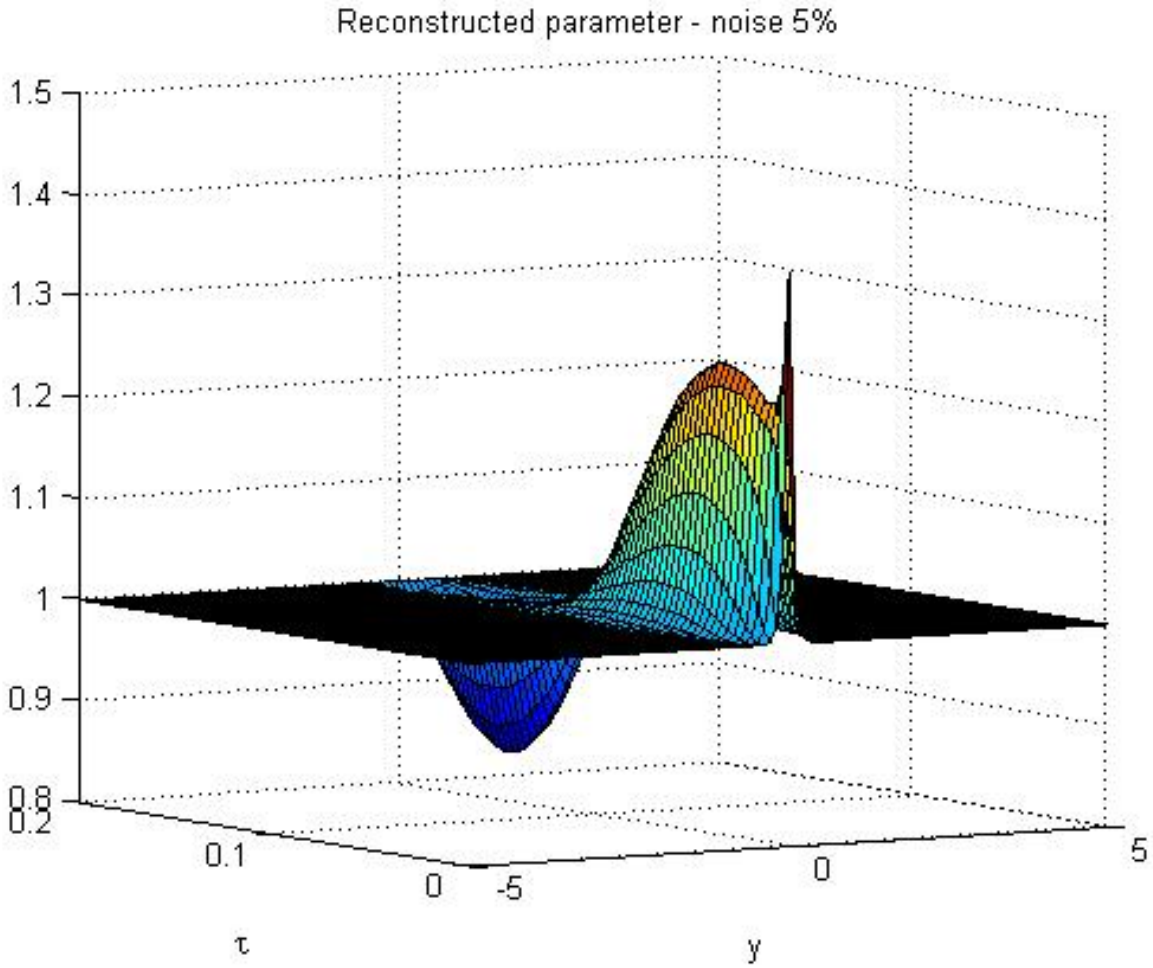
# Numerical Examples 1 - error - 4000 steps



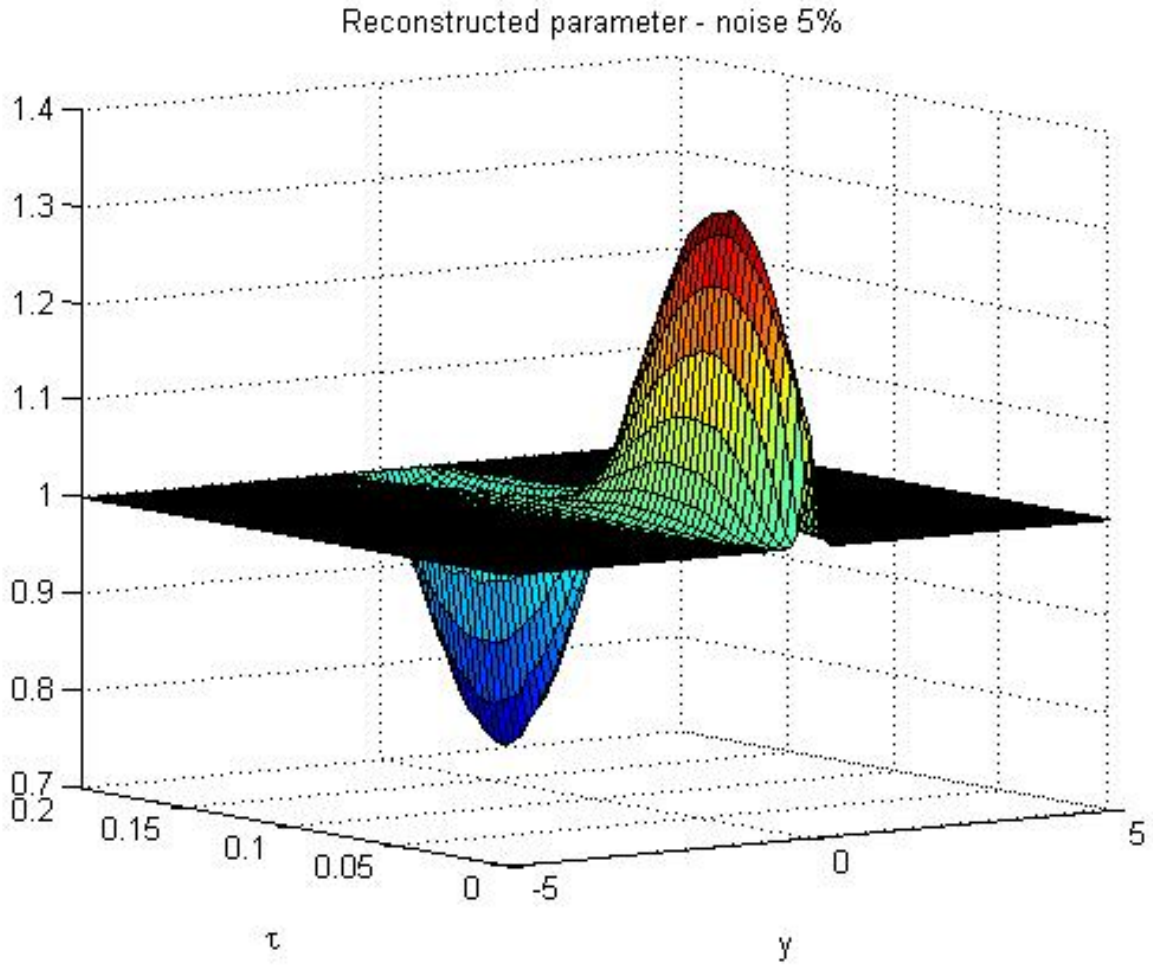

Evolution of the Log of the Residual Norm
0
-2
-4
-6
-8
0
500
1000
1500
2000
2500
3000
3500
4000
Iteration number


Evolution of the Log of the $||a_k - a^+||$
2
1
0
-1
-2
0
500
1000
1500
2000
2500
3000
3500
4000
Iteration number

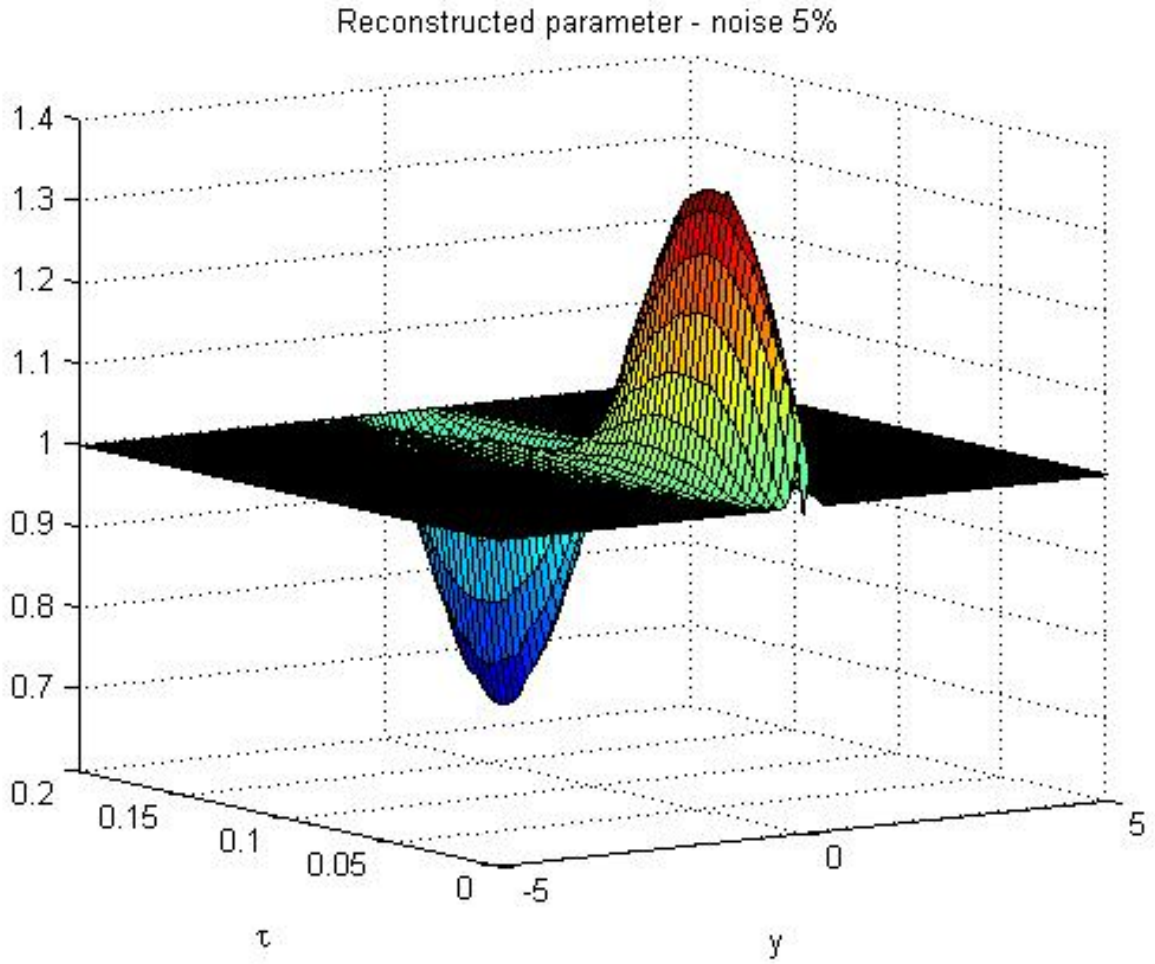
# Numerical Examples 2 - 5% noise level - 100 steps



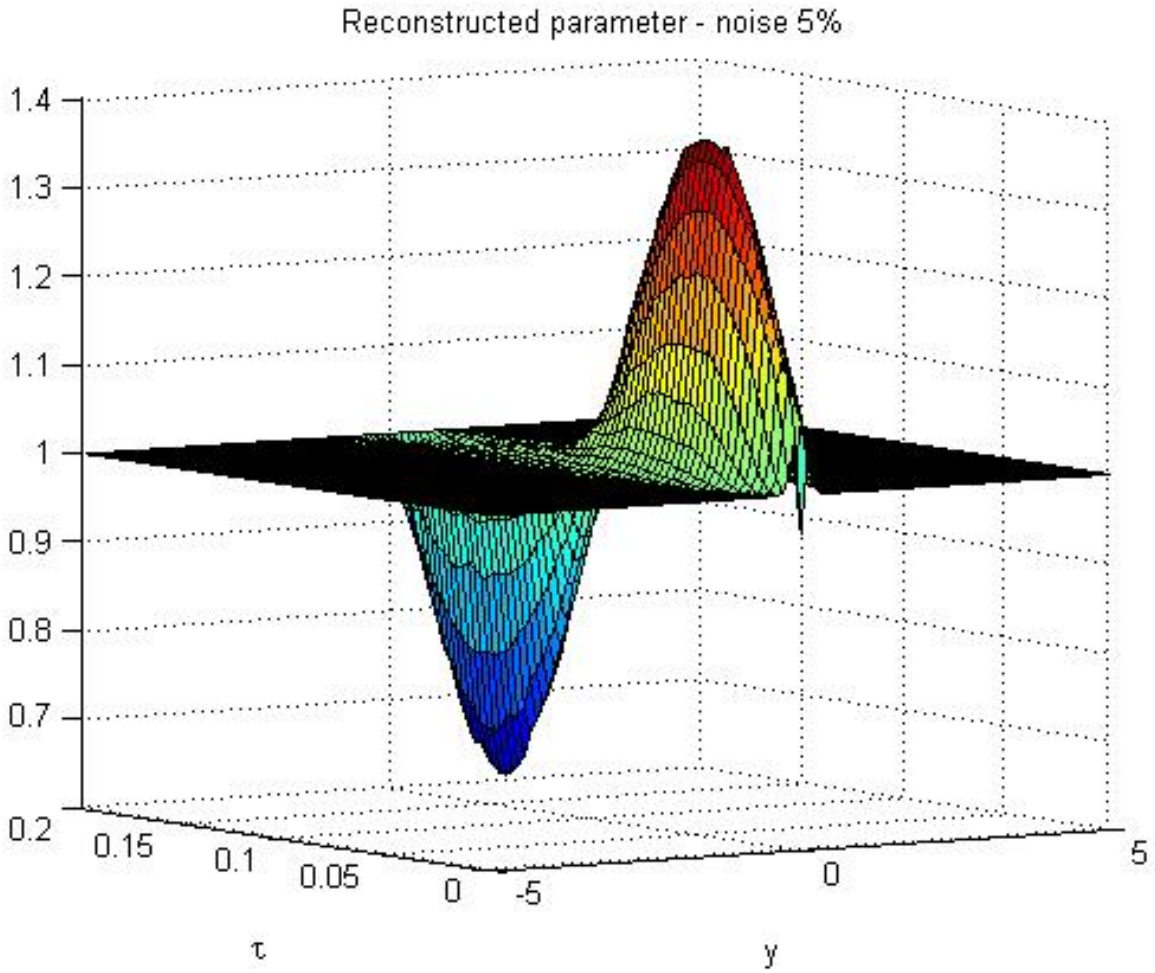
# Numerical Examples 2 - 5% noise level - 200 steps



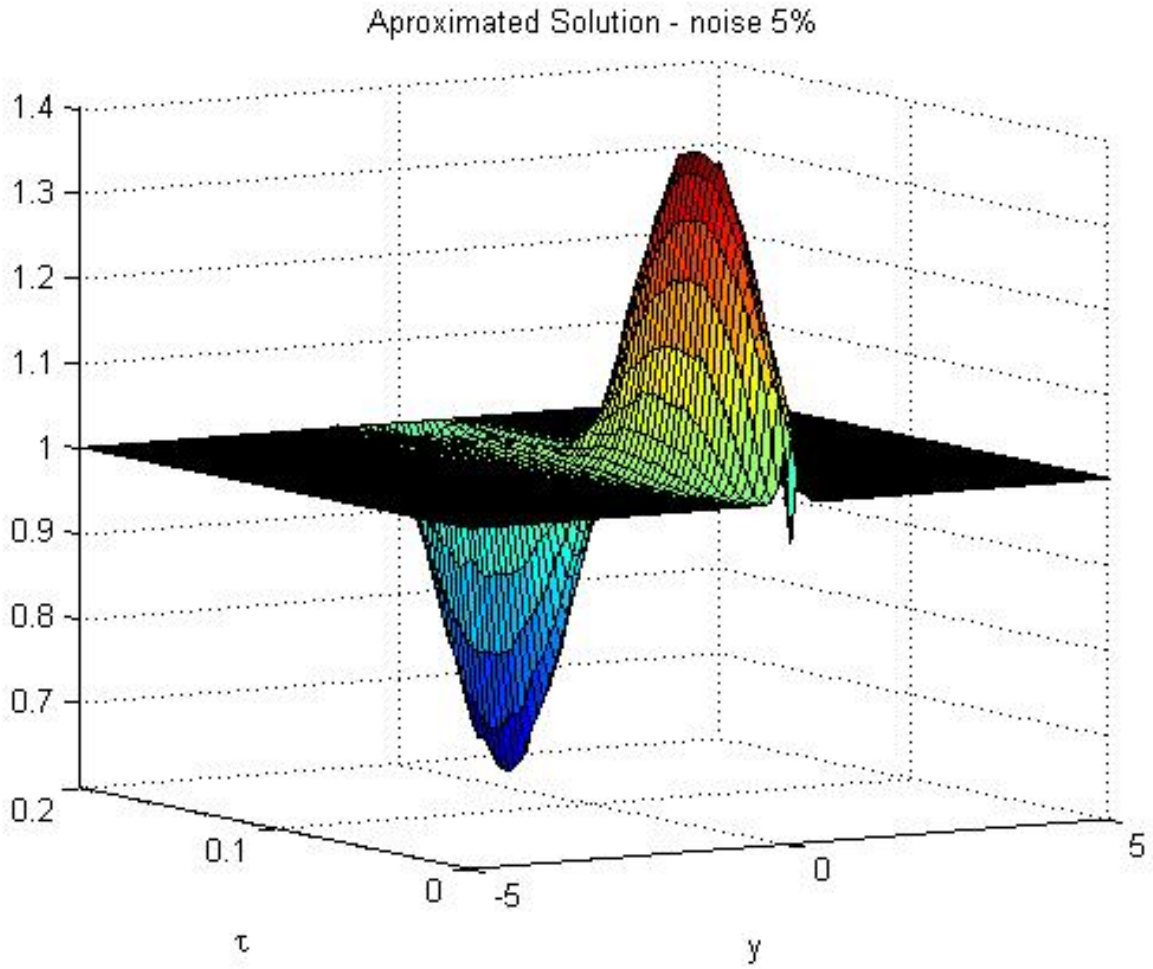
# Numerical Examples 2 - 5% noise level - 300 steps

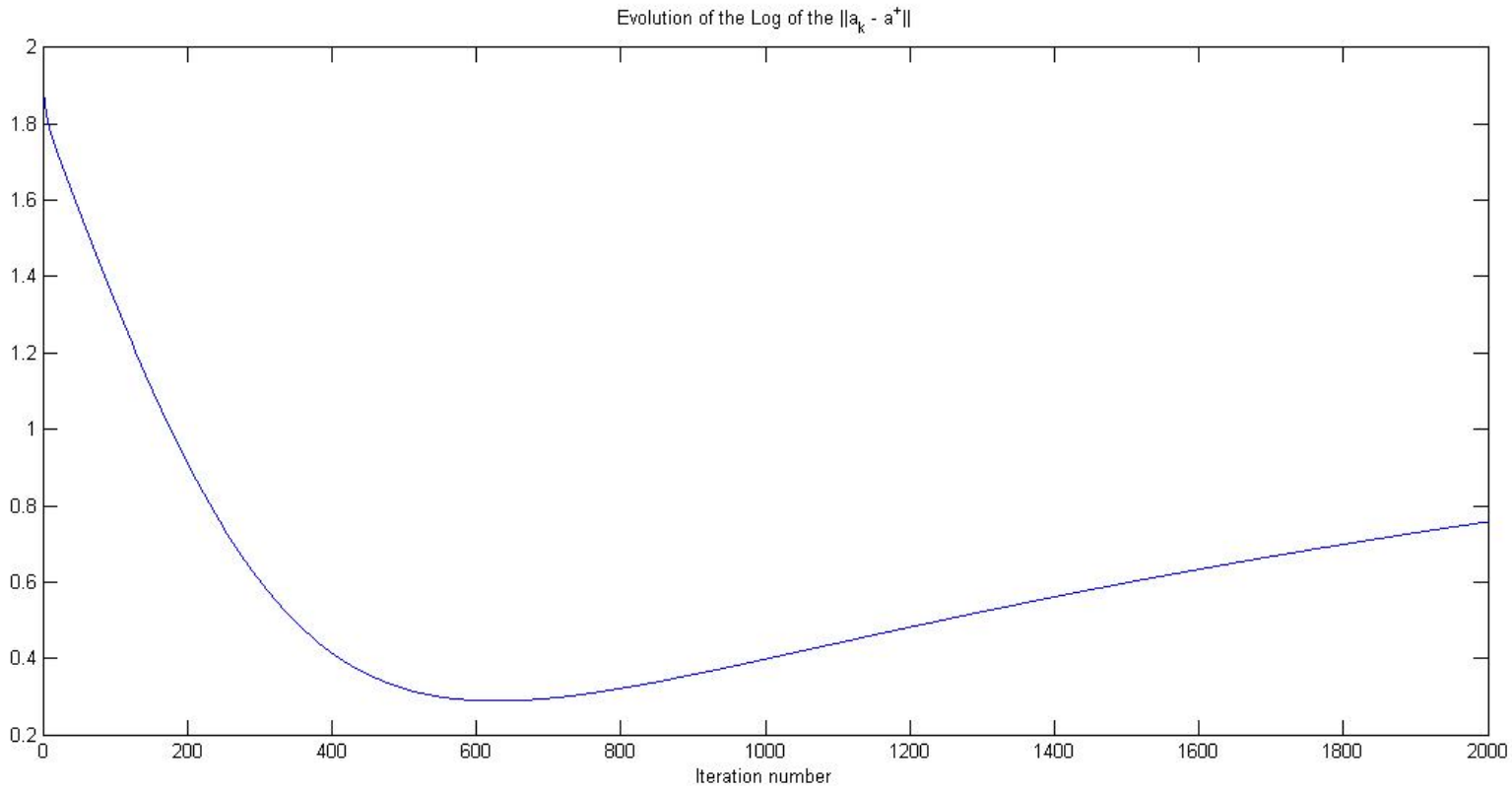


# Numerical Examples 2 - 5% noise level - 400 steps



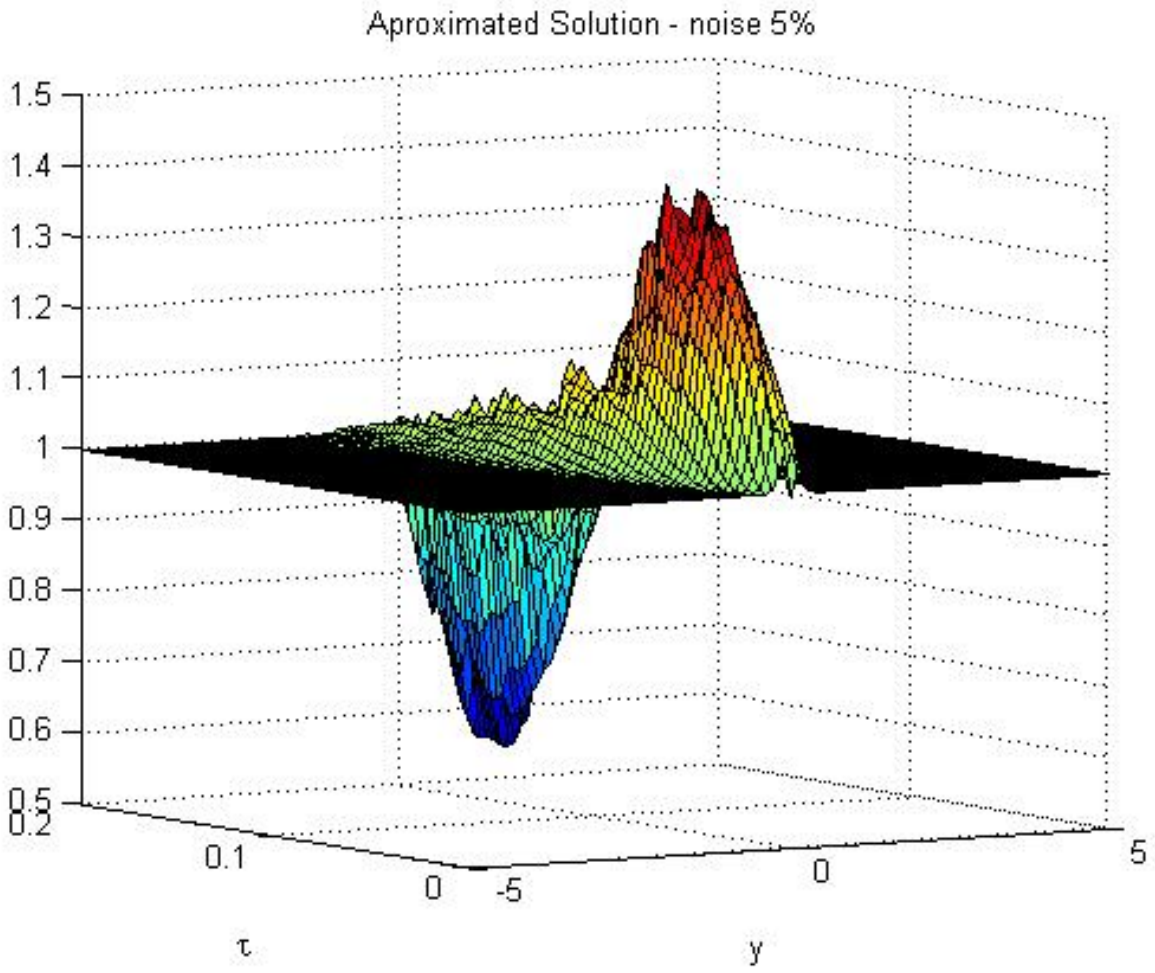
# Numerical Examples 2 - 5% noise level - Stopping criteria




Evolution of the Log of the ||a_k - a^+||
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
200
400
600
800
1000
1200
1400
1600
1800
2000
Iteration number

# Numerical Examples 2 - 5% noise level - 2000 iterations

Too many iterations!!!



# Local Vol Surface Reconstruction w/ Synthetic Data

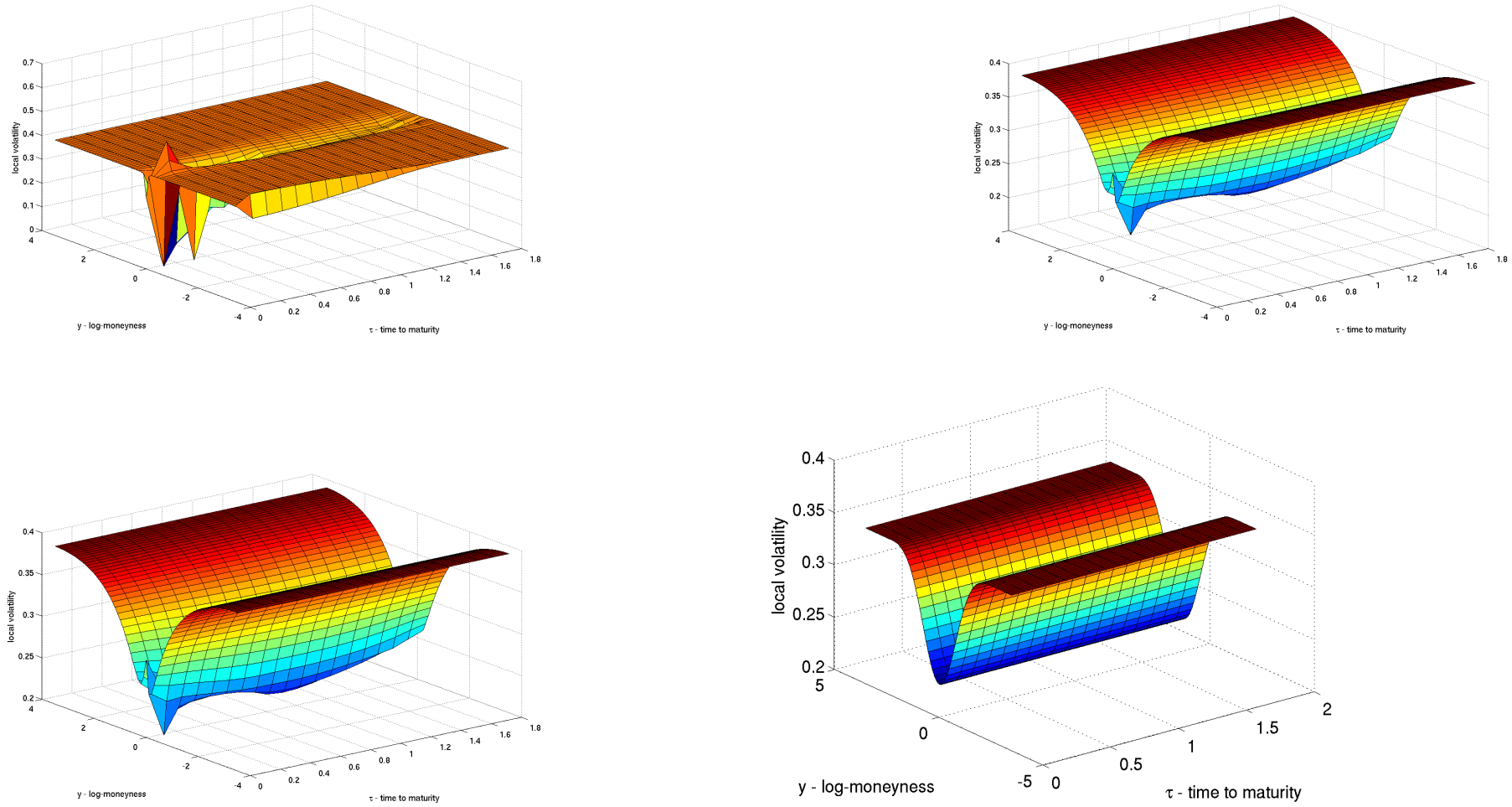



Figure: Calibration of the local volatility in 5 iterations. Shown from the upper left, clockwise, are the 1st iteration, 3rd iteration, 5th iteration and the ground truth.

# Local Vol Surface for WTI Crude Oil

totally nonparametric

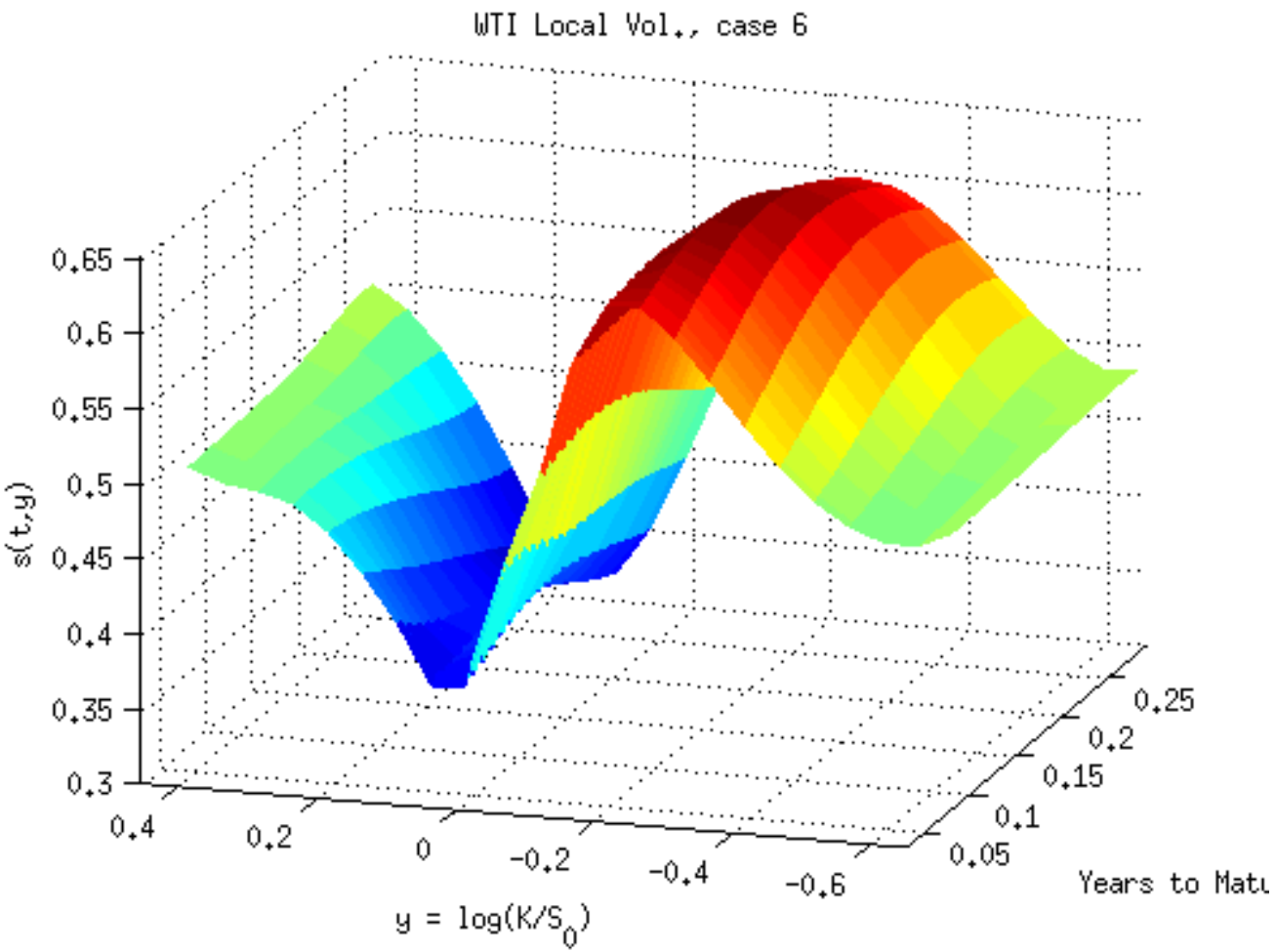



Figure: Local Vol Surface associated to Heston Model Calibrated on SPX data

# Real Data Results

## Note the scarcity of the data

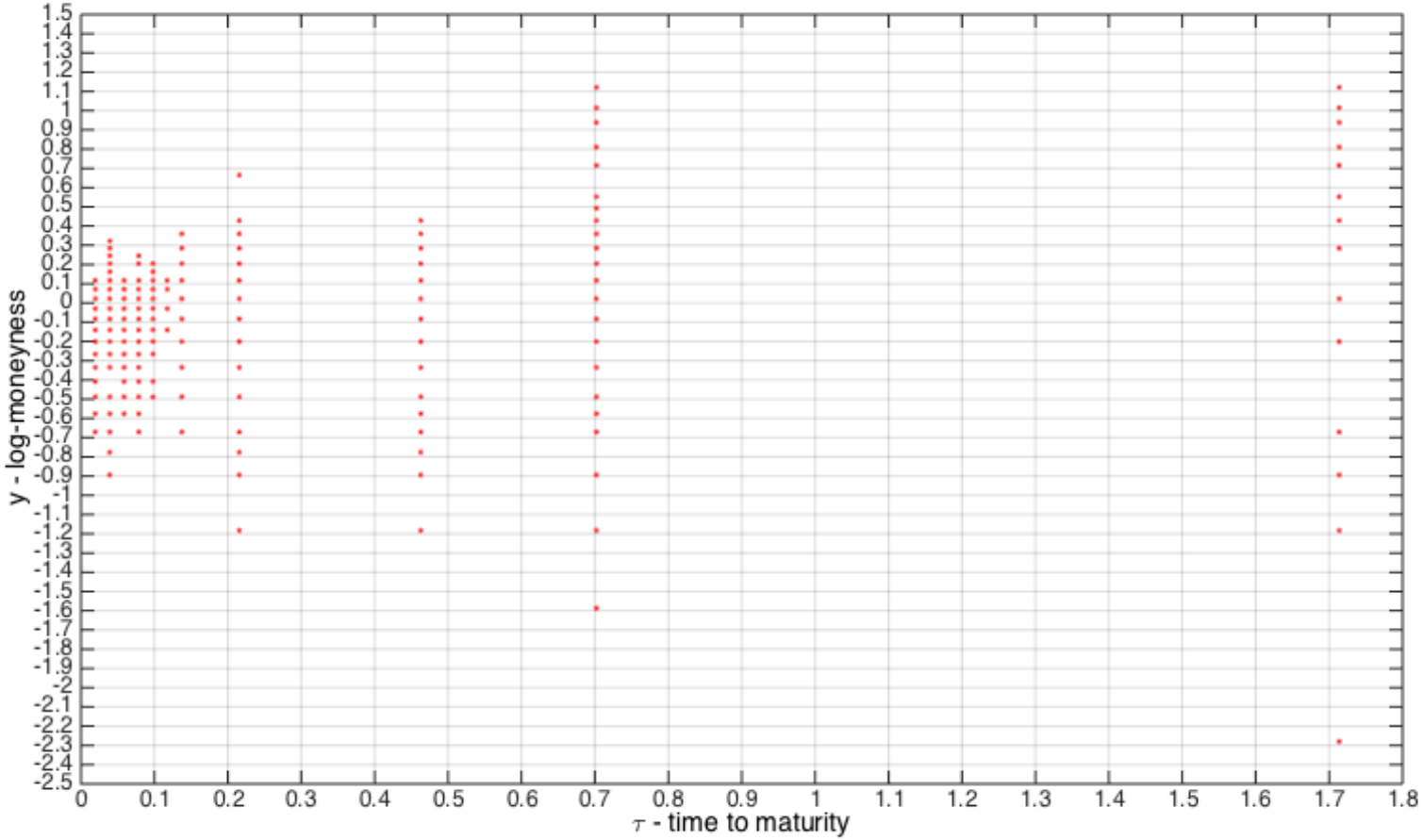



Figure: Data locations for a PBR set in the $(\tau, y)$ domain with our coarsest mesh in the background.

# Numerical Examples: with Real Data

Reconstruction of $a = \sigma^2/2$ with PBR Stock Data (implemented by Vinicius L. Albani/IMPA)

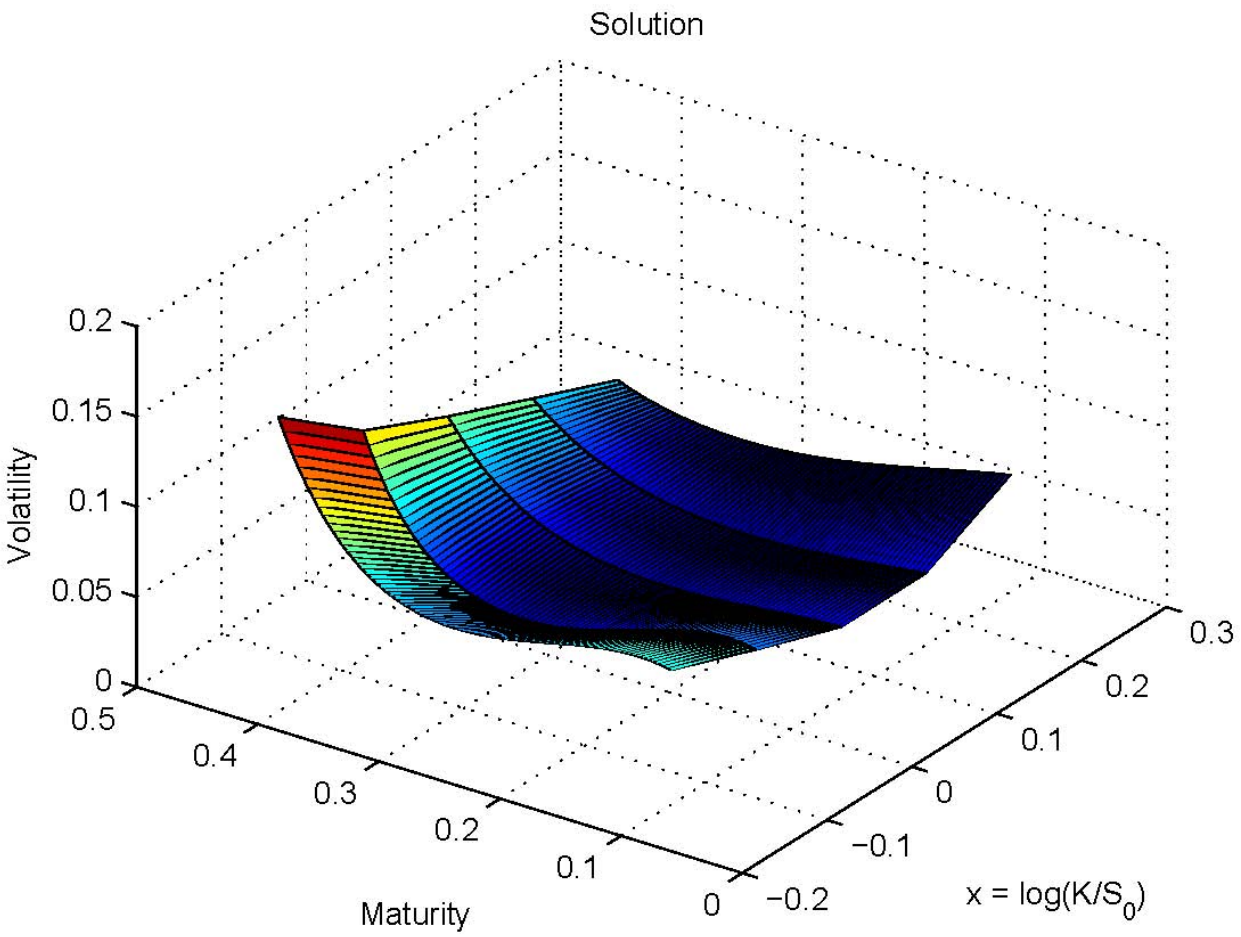



Figure: Minimal Entropy functional / Landweber Method / a priori Implied Vol / maturities: 2010-11

# Local Vol Surface for Henry Hub Natural Gas

In the next plots we show an *online* approach (joint work w/ V. Albani). We performed the following:

- We consider the evolution of prices of futures and options for several days but *kept* the maturity dates and all the other features of the options.
- Calibrated using the extra information.
- This is part of an extension of the above results that leads to incorporating the flow of information.

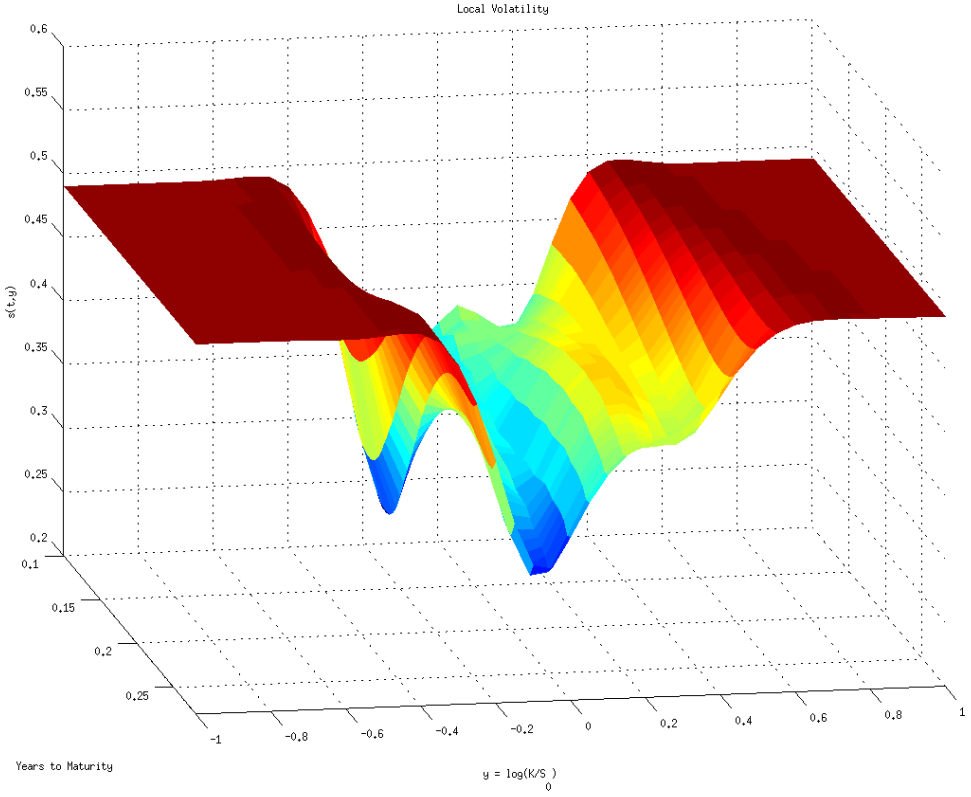



Figure: Local Vol Surface associated to Henry Hub Gas Prices

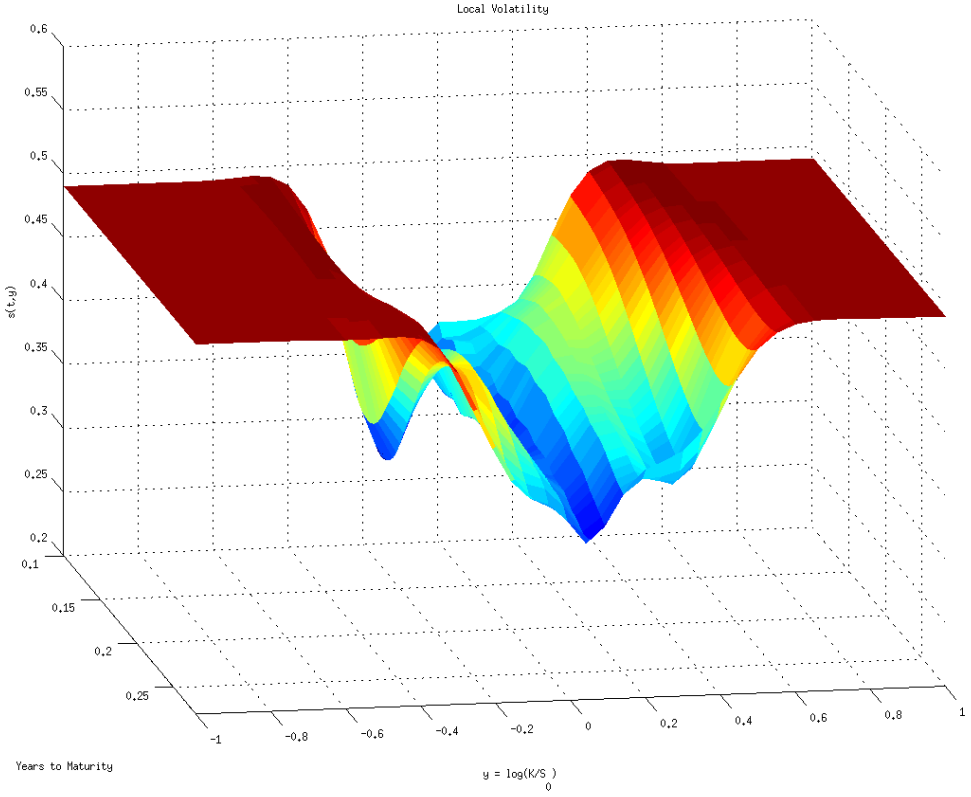



Figure: Local Vol Surface associated to Henry Hub Gas Prices

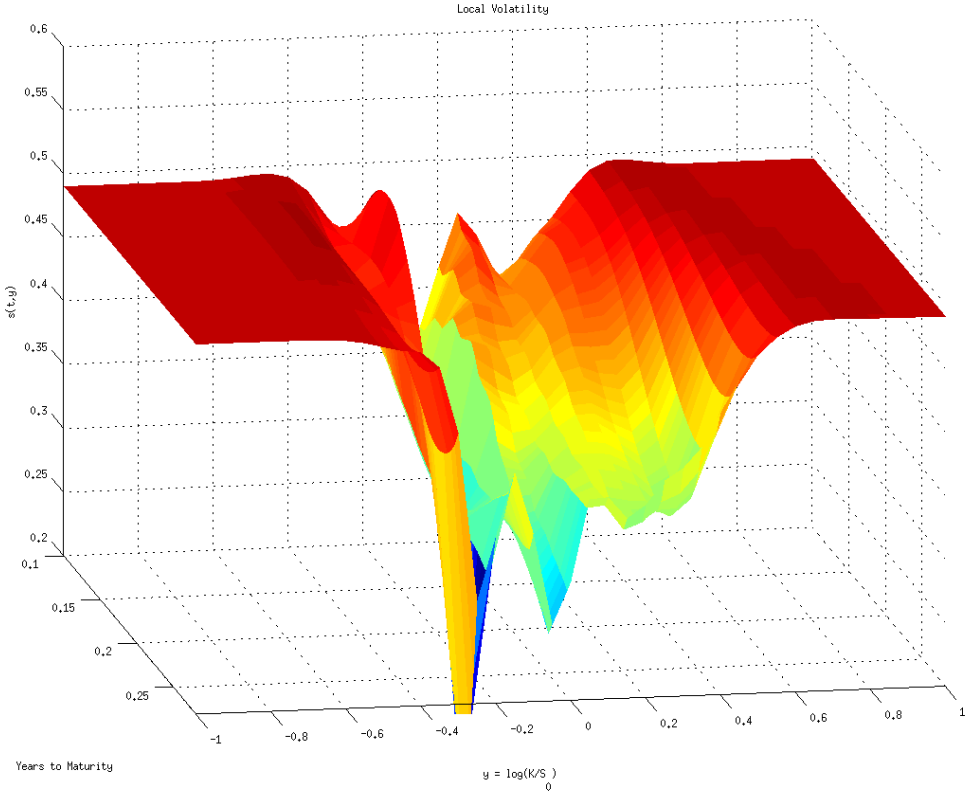



Figure: Local Vol Surface associated to Henry Hub Gas Prices

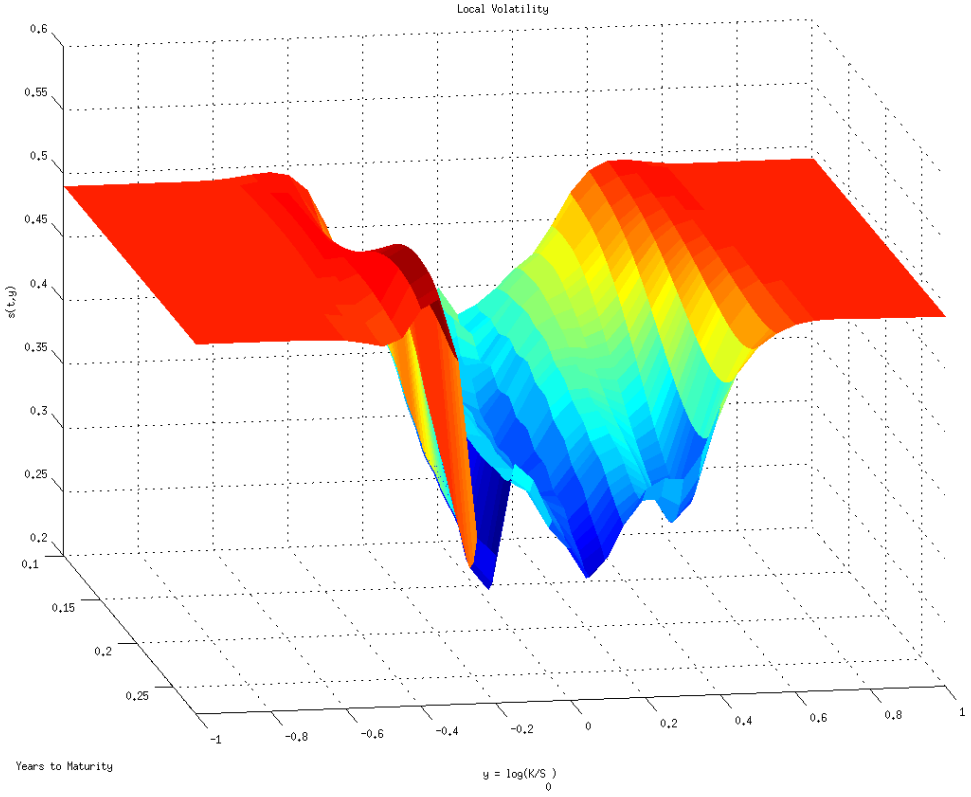



Figure: Local Vol Surface associated to Henry Hub Gas Prices

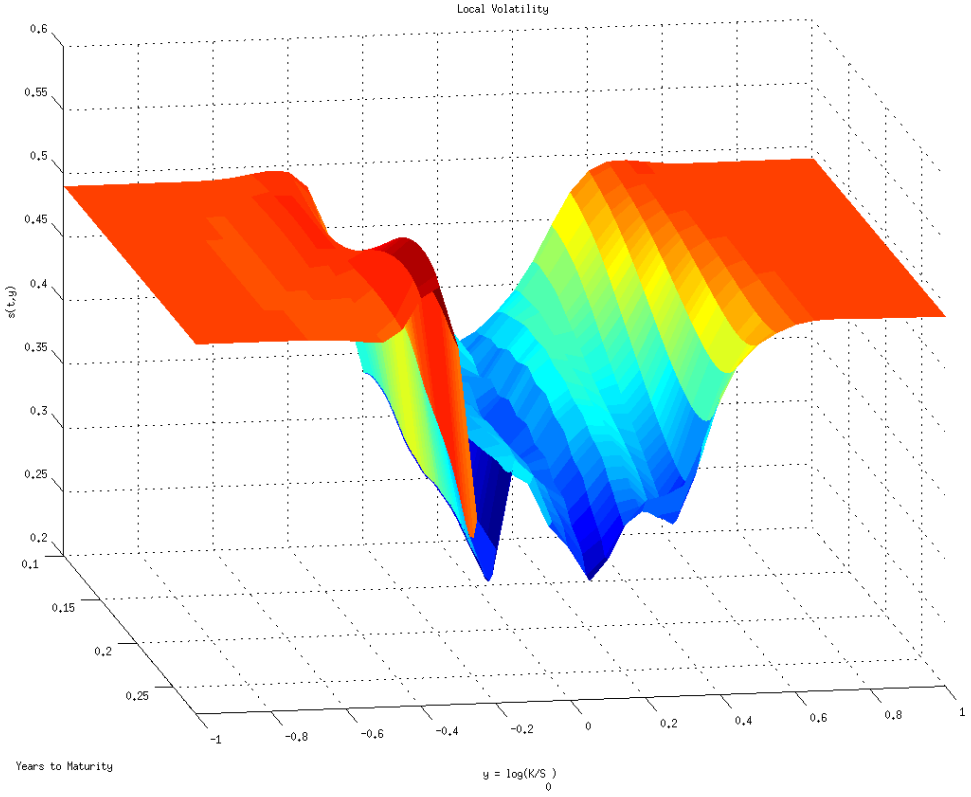



Figure: Local Vol Surface associated to Henry Hub Gas Prices

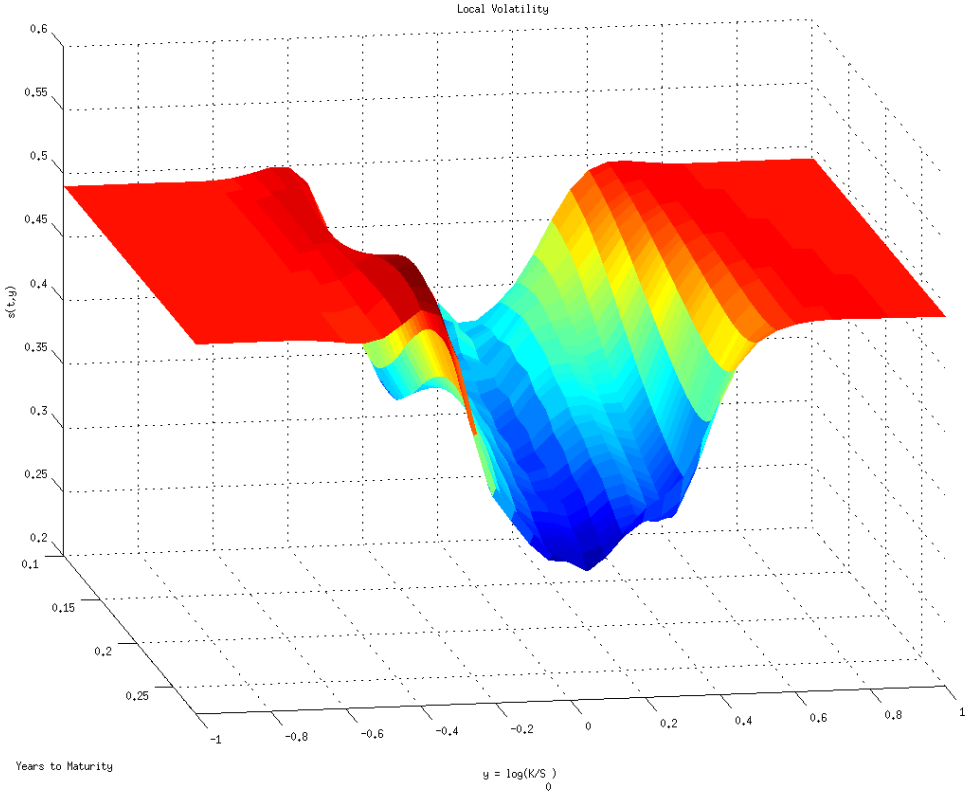



Figure: Local Vol Surface associated to Henry Hub Gas Prices

# Calibration of the Local Volatility Surface

## Adherence of the Implied Volatility

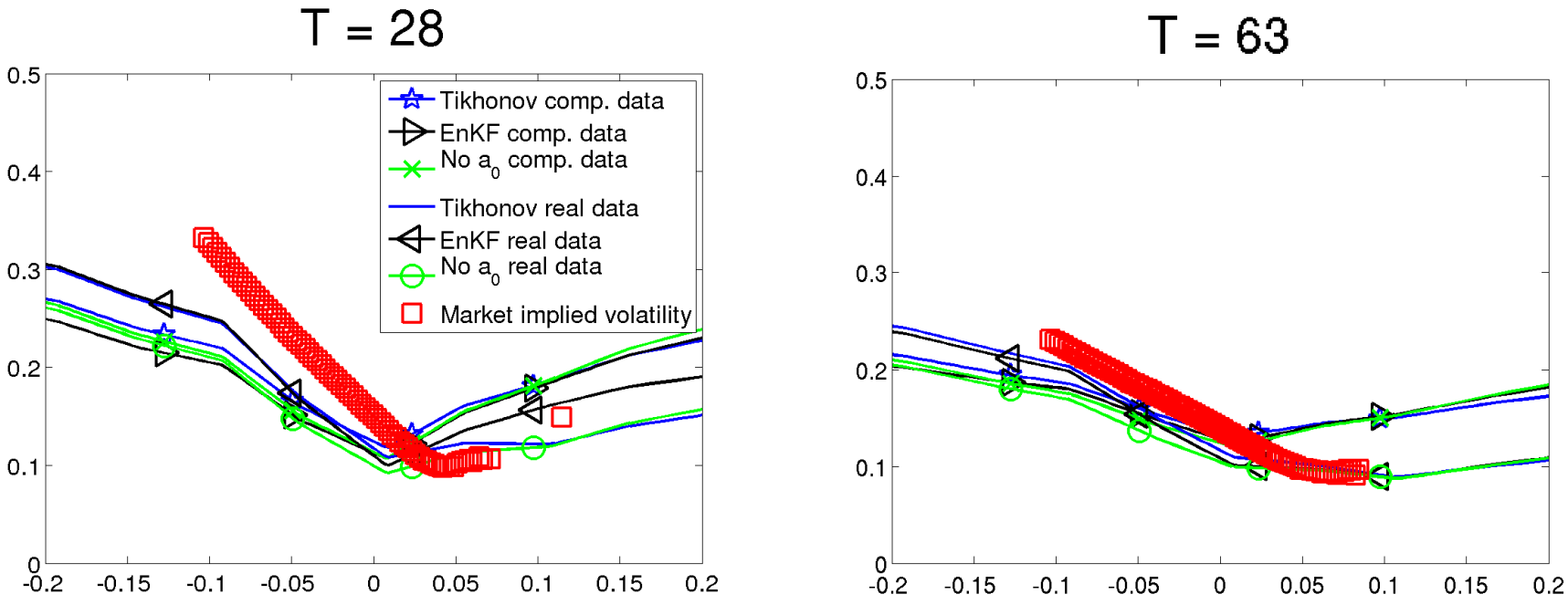



Figure: Implied (Black-Scholes) volatility corresponding to the local volatility surfaces obtained with the six method variants compared to the market one SPX data.

# Calibration of the Local Volatility Surface

## Adherence of the Implied Volatility (cont.1)

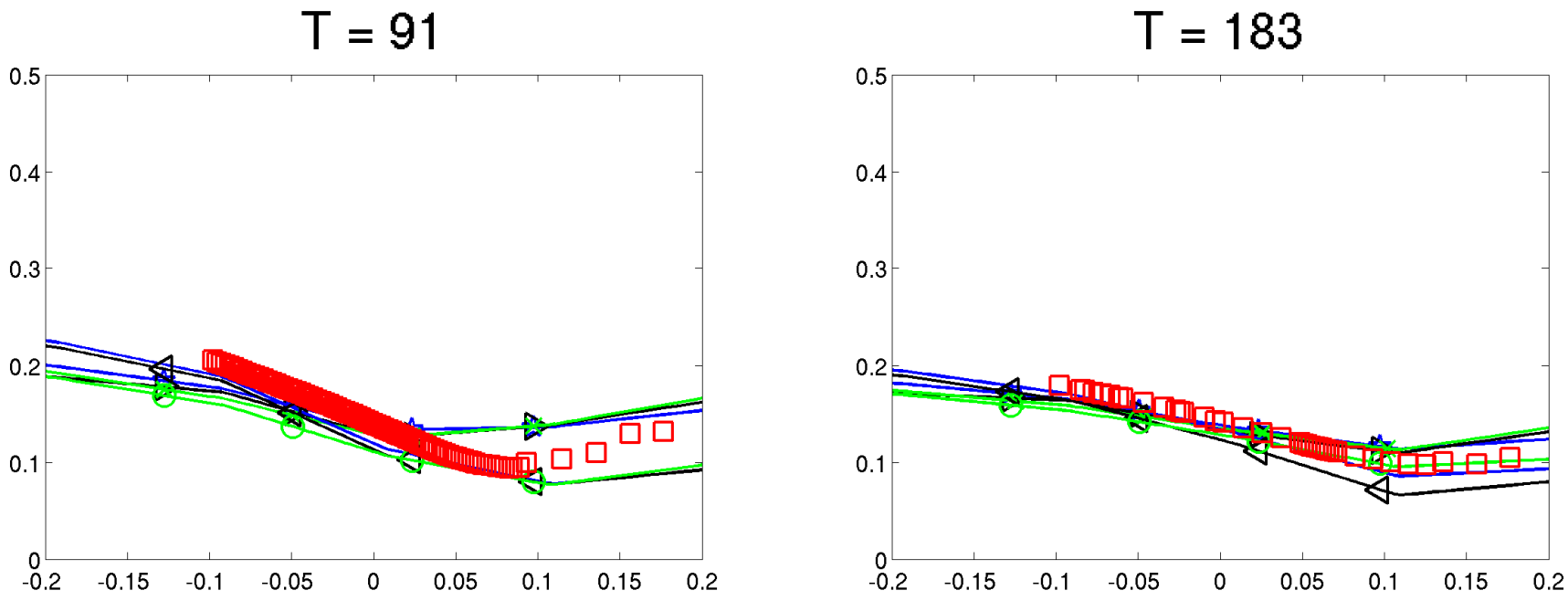



Figure: Implied (Black-Scholes) volatility corresponding to the local volatility surfaces obtained with the six method variants compared to the market one SPX data.

# Calibration of the Local Volatility Surface

## Adherence of the Implied Volatility (cont.2)

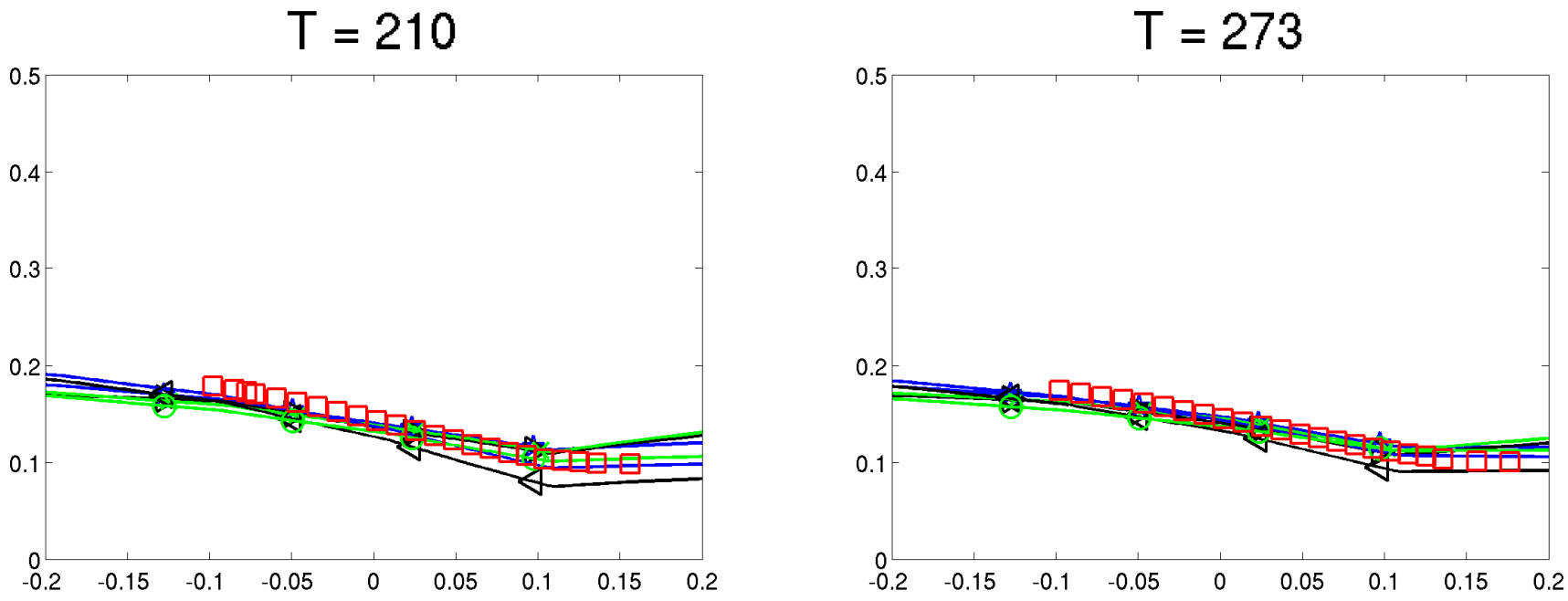



Figure: Implied (Black-Scholes) volatility corresponding to the local volatility surfaces obtained with the six method variants compared to the market one SPX data.

# Calibration of the Local Volatility Surface

## Adherence of the Implied Volatility (cont.3)

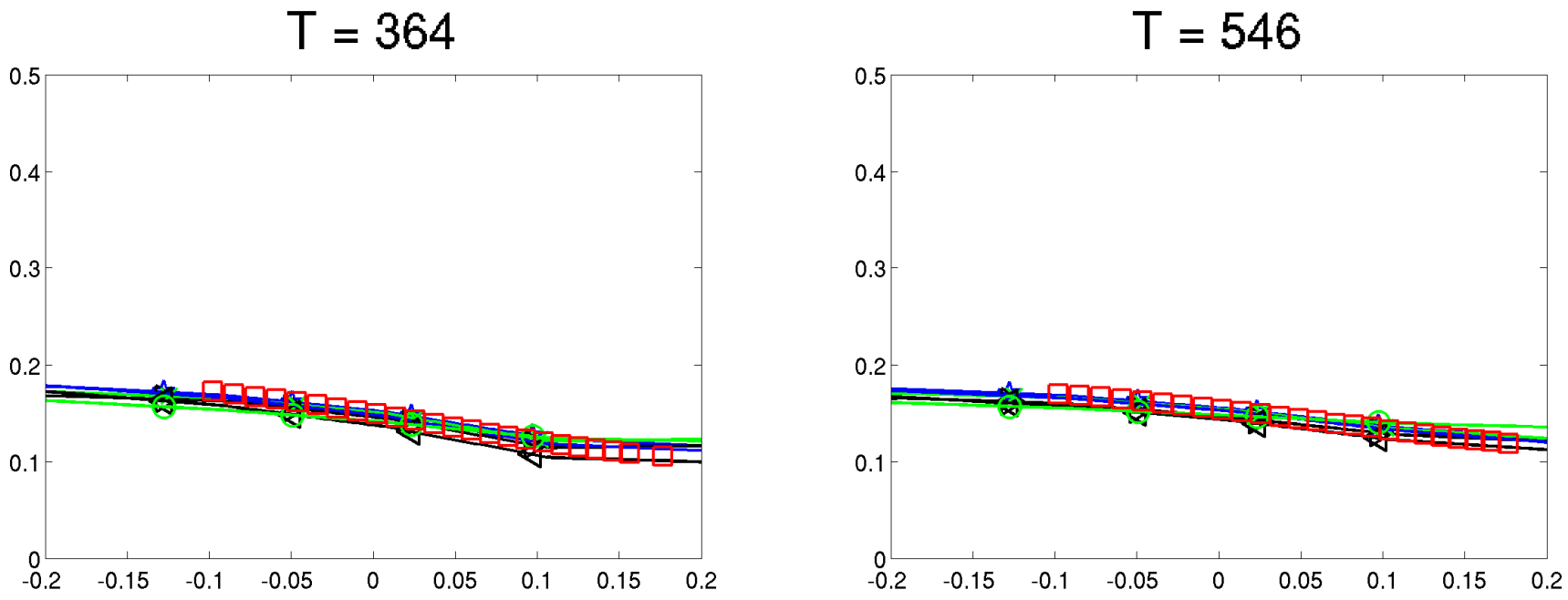



Figure: Implied (Black-Scholes) volatility corresponding to the local volatility surfaces obtained with the six method variants compared to the market one SPX data.

# Conclusions

- Volatility surface calibration is a classical and fundamental problem.
- We developed a unifying framework for the regularization that makes use of tools from Inverse Problem theory and Convex Analysis and established:
  1. Convergence of the regularized sol. w.r.t the noise level in different topologies
  2. Implemented a Landweber type calibration algorithm.
  3. Implemented an Ensemble Kalman Filter algorithm.
- Extended the theory and the algorithms to commodity derivatives.
- Developed an *Online Calibration Methodology*
- Future Possibilities:
  1. Incorporate another source of stochasticity (generalized Heston models)
  2. Integrate with the evaluation of complex derivatives

.





**Collaborators:**

V. Albani (IMPA), A. de Cezaro (FURG), O. Scherzer (Vienna), U. Ascher (UBC), X. Yang (IMPA).

**THANK YOU FOR YOUR ATTENTION!!!**

# Announcement of a Conference in Math Finance

## Research in Options RiO 2015 - IMPA, Rio de Janeiro Nov 29th - Dec 5th, 2015

**WEBSITE IN CONSTRUCTION!:** www.impa.br/˜ zubelli/rio2016


Some Regulars ...

- M. Avellaneda (CO-ORGANIZER)
- Raphael Douady
- Bruno Dupire (CO-ORGANIZER!)
- Marco Fritelli
- Matheus Grasselli
- Lane Hughston
- Roger Lee
- Chris Rogers
- YOUR NAME COULD BE HERE!!!

# The 2010 Version

M. Avellaneda, R. Buff, C. Friedman, N. Grandchamp, L. Kruk, and J. Newman.
Weighted Monte Carlo: A new technique for calibrating asset-pricing models.
Spigler, Renato (ed.), Applied and industrial mathematics, Venice-2, 1998. Selected papers from the 'Venice-2/Symposium', Venice, Italy, June 11-16, 1998. Dordrecht: Kluwer Academic Publishers. 1-31 (2000)., 2000.

M. Avellaneda, C. Friedman, R. Holmes, and D. Samperi.
Calibrating volatility surfaces via relative-entropy minimization.
*Appl. Math. Finance*, 4(1):37–64, 1997.

M. Avellaneda.
Minimum-relative-entropy calibration of asset-pricing models.
*International Journal of Theoretical and Applied Finance*, 1(4):447–472, 1998.

Marco Avellaneda.
The minimum-entropy algorithm and related methods for calibrating asset-pricing model.

In *Trois applications des mathématiques*, volume 1998 of *SMF Journ. Annu.*, pages 51–86. Soc. Math. France, Paris, 1998.

Marco Avellaneda.
The minimum-entropy algorithm and related methods for calibrating asset-pricing models.
In *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*, number Extra Vol. III, pages 545–563 (electronic), 1998.

I. Bouchouev and V. Isakov.
The inverse problem of option pricing.
*Inverse Problems*, 13(5):L11–L17, 1997.

James N. Bodurtha, Jr. and Martin Jermakyan.
Nonparametric estimation of an implied volatility surface.
*Journal of Computational Finance*, 2(4), Summer 1999.

S. Crépey.
Calibration of the local volatility in a generalized Black-Scholes model using Tikhonov regularization.
*SIAM J. Math. Anal.*, 34(5):1183–1206 (electronic), 2003.

A. De Cezaro, O. Scherzer, and J. P. Zubelli.
Convex regularization of local volatility models from option prices: convergence analysis and rates.
*Nonlinear Anal.*, 75(4):2398–2415, 2012.

Emanuel Derman, Iraj Kani, and Joseph Z. Zou.
The local volatility surface: Unlocking the information in index option prices.
*Financial Analysts Journal*, 52(4):25–36, 1996.

B. Dupire.
Pricing with a smile.
*Risk*, 7:18– 20, 1994.

H. Egger and H. W. Engl.
Tikhonov regularization applied to the inverse problem of option pricing: convergence analysis and rates.
*Inverse Problems*, 21(3):1027–1045, 2005.

H. W. Engl, M. Hanke, and A. Neubauer.

*Regularization of inverse problems*, volume 375 of *Mathematics and its Applications*.
Kluwer Academic Publishers Group, Dordrecht, 1996.

B. Hofmann and R. Krämer.
On maximum entropy regularization for a specific inverse problem of option pricing.
*J. Inverse Ill-Posed Probl.*, 13(1):41–63, 2005.

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer.
A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators.
*Inverse Problems*, 23(3):987–1010, 2007.