Equilibrium Routing under Uncertainty

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Stochastic Mathematical Optimization and Variational Analysis

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Models to describe traffic flows under congestion

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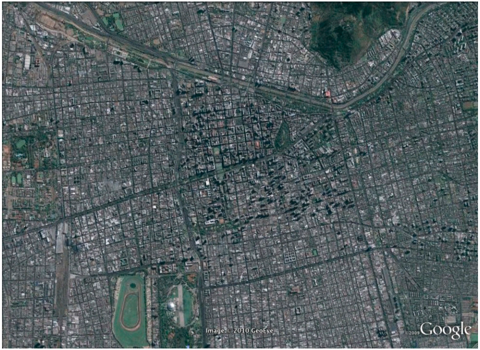
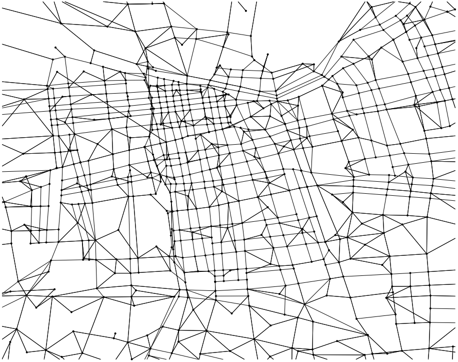


SANTIAGO
6.000.000 people
11.000.000 daily trips
1.750.000 car trips

Morning peak
500.000 car trips
29.000 OD pairs

Models to describe traffic flows under congestion



2266 nodes / 7636 arcs / 409 centroids

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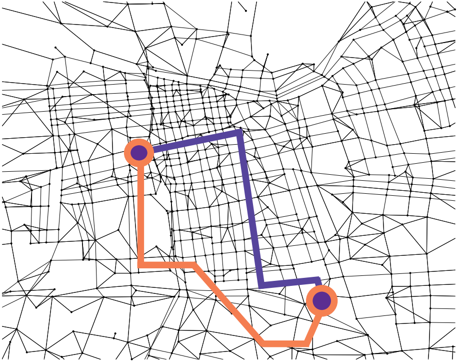

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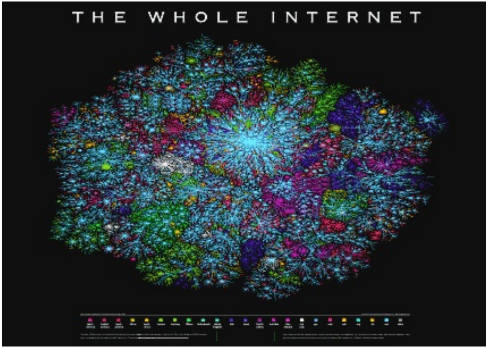
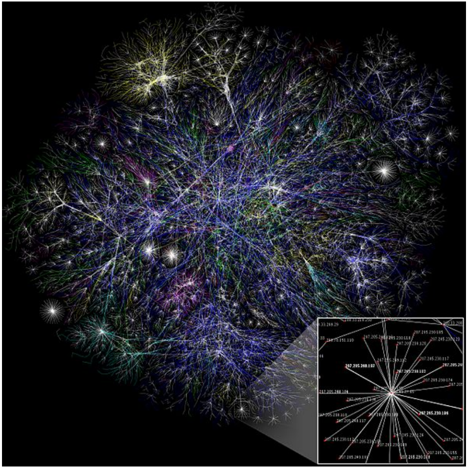
Question: control traffic flows and congestion



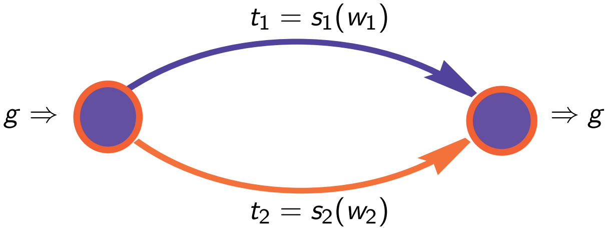
INTERNET	
294.000.000.000	mails/day
2.000.000.000	videos/day
8.500.000.000	webpages
2.100.000.000	users

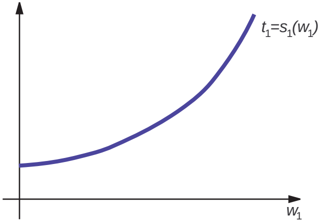
Question: control traffic flows and congestion



INTERNET Backbone
193.000.000 domains
75.000.000 servers
35.000 AS's

Equilibrium: Wardrop's basic idea... 1952





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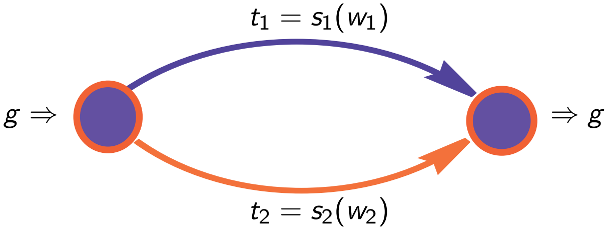
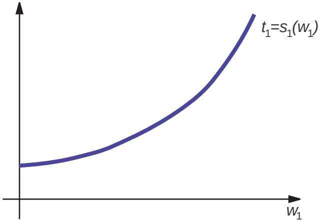
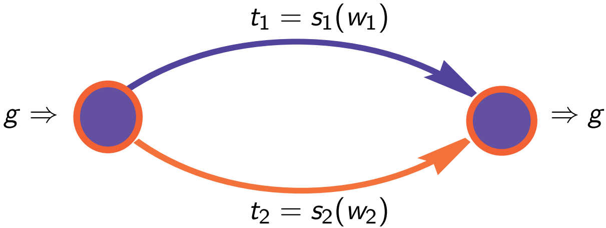
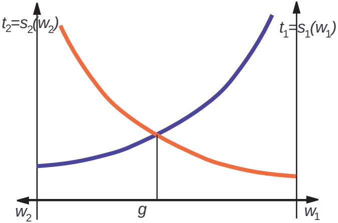




$$\begin{cases} g = w_1 + w_2 \\ w_1 > 0 \Rightarrow t_1 \leq t_2 \\ w_2 > 0 \Rightarrow t_2 \leq t_1 \end{cases}$$

Equilibrium: Wardrop's basic idea... 1952





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Outline

1. Equilibrium models
2. Adaptive learning
3. TCP/IP protocols
4. Risk-averse routing

Deterministic & stochastic equilibrium models

Wardrop Equilibrium (Wardrop'52)

$$\text{Given} \begin{cases} \text{network} & (N, A) \\ \text{arc travel times} & t_a = s_a(w_a) \\ \text{travel demands} & g_i^d \geq 0 \\ \text{routes} & \mathcal{R}_i^d \end{cases}$$

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Split $g_i^d = \sum_{r \in \mathcal{R}_i^d} x_r$ with $x_r \geq 0$ so that only shortest routes are used

$$\boxed{x_r > 0 \;\Rightarrow\; T_r = \tau_i^d}$$

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$$\boxed{x_r > 0 \;\Rightarrow\; T_r = \tau_i^d}$$

where

$$\begin{aligned} \tau_i^d &= \min_{r \in \mathcal{R}_i^d} T_r && \text{(minimal time)} \\ T_r &= \textstyle\sum_{a \in r} s_a(w_a) && \text{(route times)} \\ w_a &= \textstyle\sum_{r \ni a} x_r && \text{(total arc flows)} \end{aligned}$$

Variational characterization (Beckman-McGuire-Winsten'56)

Theorem

$(w_a^*)_{a\in A}$ *Wardrop equilibrium* $\Leftrightarrow$ *optimal solution of*

$$(P) \quad \begin{cases} \text{Min} & \displaystyle\sum_a \int_0^{w_a} s_a(z)\,dz \\ \textit{s.t. flow conservation} & \end{cases}$$

Proof

$r \in \mathcal{R}_i^d, x_r > 0 \Rightarrow T_r = \min_{p \in \mathcal{R}_i^d} T_p$ is equivalent to

$$\sum_{(i,d)} \sum_{r \in \mathcal{R}_i^d} T_r(\tilde{x}_r - x_r) \geq 0 \qquad \text{for all feasible } \tilde{x}$$

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Exchanging the order of summation this becomes

$$\sum_{a \in A} \sum_{(i,d)} \sum_{r \in \mathcal{R}_i^d, r \ni a} s_a(w_a)(\tilde{x}_r - x_r) \geq 0 \qquad \text{for all feasible } \tilde{x}$$

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Exchanging the order of summation this becomes

$$\sum_{a \in A} s_a(w_a)(\tilde{w}_a - w_a) \geq 0 \qquad \text{for all feasible } \tilde{x}$$

which are precisely the optimality conditions for the convex program

$$\min_{w \text{ feasible}} \sum_{a \in A} \int_0^{w_a} s_a(z) dz$$

□

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Corollary

1. *There exists a Wardrop equilibrium* w^*
2. *Equilibrium travel times* $t_a^* = s_a(w_a^*)$ *are unique*
3. *If* $s_a(\cdot)$ *strictly increasing* $\Rightarrow$ w^* *unique*

Dual characterization (Fukushima'84)

Change of variables: $w_a \leftrightarrow t_a$

$$(D) \qquad \operatorname{Min}_{t} \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(z)\,dz - \sum_{i,d} g_i^d \tau_i^d(t)}_{\substack{\phi(t) \\ \text{strictly convex}}}$$

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$$t \mapsto \tau_i^d(t) \quad = \quad \begin{array}{l} \text{minimum travel time} \\ \text{concave, non-smooth, polyhedral} \end{array}$$

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concave, non-smooth, polyhedral

$$\boxed{\tau_i^d = \min_{a \in A_i^+} [t_a + \tau_{j_a}^d]}$$

Bellman's equations

Method of Successive Averages

Algorithm 1 MSA - main iteration

1: Compute $t_a^n = s_a(w_a^n)$
2: Assign g_i^d to shortest routes
3: Compute arc flows $\tilde{w}_a^n = \Phi_a(w^n)$
4: Update $w^{n+1} = (1-\alpha_n)w^n + \alpha_n \tilde{w}^n$

Wardrop equilibrium $\equiv$ Fixed point of Φ

What if travel times are uncertain?

Copenhagen – DTU Transport (www.transport.dtu.dk)

Figure 2: Example of real time illustration of congestion (Source: Vejdirektoratet, www.trafikken.dk)

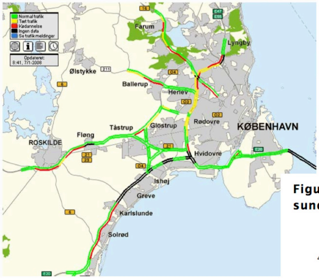
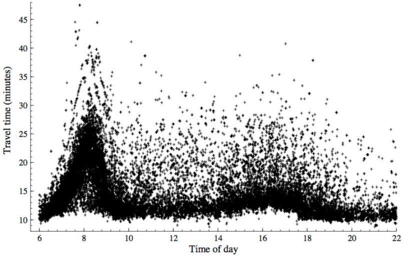


Figure 7: Observations of travel time by time of day. Frederiks-sundsvej, inward direction



Stochastic User Equilibrium (Dial'71, Fisk'80)

Drivers have different perceptions of route costs

$$\left.\begin{aligned} \tilde{T}_r &= T_r + \epsilon_r \\ \tilde{\tau}_i^d &= \min_{r\in\mathcal{R}_i^d} \tilde{T}_r \end{aligned}\right\} \text{ random variables}$$

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Demand splits according to the pbb of each route being optimal

$$\boxed{x_r = g_i^d \, \mathbb{P}(\tilde{T}_r = \tilde{\tau}_i^d)}$$

with $t_a = s_a(w_a)$ and $w_a = \sum_{r \ni a} x_r$ as before

LOGIT MODEL (Dial'71, Fisk'80)

ϵ_r i.i.d. Gumbel noise (supported by Gnedenko's theorem)

$$x_r = g_i^d \frac{\exp(-\beta T_r)}{\sum_{s \in \mathcal{R}_i^d} \exp(-\beta T_s)}$$

Drawbacks: independence is unlikely & tractable only for small networks

PROBIT MODEL (Daganzo'82)

ϵ_r correlated Normal noise

No closed form equations $\Rightarrow$ Montecarlo

Drawback: tractable only for very small networks

Discrete choice models

Finite set of alternatives $i \in I$ with random costs $\tilde{z}_i = z_i + \varepsilon_i$.

Choose alternative of minimum cost. The *expected cost* is

$$\varphi(z) = \mathbb{E}[\min_{i \in I}(z_i + \varepsilon_i)]$$

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Proposition

1. *φ is a concave finite function*
2. *If $(\varepsilon_i)_{i \in I}$ has continuous distribution then φ is smooth with*

$$\mathbb{P}(z_i + \varepsilon_i \text{ optimal}) = \frac{\partial \varphi}{\partial z_i}$$

EXAMPLE: Multinomial Logit, $\varepsilon_k \sim$ i.i.d. Gumbel

$$\varphi(z) = -\frac{1}{\beta}\ln[\sum_j \exp(-\beta z_j)]$$

$$\frac{\partial \varphi}{\partial z_k} = \frac{\exp(-\beta z_k)}{\sum_j \exp(-\beta z_j)}$$

Dual characterization of SUE

$$(D) \qquad \min_{t} \underbrace{\sum_{a} \int_{0}^{t_a} s_a^{-1}(z)\, dz - \sum_{i,d} g_i^d \tau_i^d(t)}_{\substack{\phi(t) \\ \text{strictly convex}}}$$

Dual characterization of SUE

$$(D) \qquad \underset{t}{\text{Min}} \quad \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(z)\, dz - \sum_{i,d} g_i^d \tau_i^d(t)}_{\substack{\phi(t) \\ \text{strictly convex}}}$$

$$t \mapsto \tau_i^d(t) \quad = \quad \text{expected minimum travel time}$$
concave, smooth

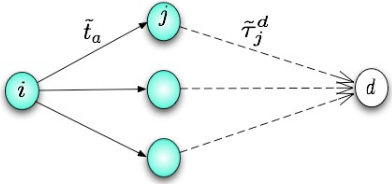
$$\boxed{\tau_i^d(t) = \mathbb{E}[\min_{r \in \mathcal{R}_i^d} T_r + \varepsilon_r]}$$

Markovian Traffic Equilibrium (Akamatsu'00, Baillon-C'06)

Routing as a stochastic dynamic programming process

$$\left.\begin{aligned} \tilde{t}_a &= t_a + \epsilon_a \\ \tilde{T}_r &= \textstyle\sum_{a\in r} \tilde{t}_a \\ \tilde{\tau}_i^d &= \min_{r\in\mathcal{R}_i^d} \tilde{T}_r \end{aligned}\right\} \text{random variables}$$

At every intermediate node i, users select a *random optimal arc*



$$\underset{a\in A_i^+}{\text{argmin}}\ \tilde{t}_a + \tilde{\tau}_{j_a}^d$$

$\Rightarrow$ Markov chain for each destination d

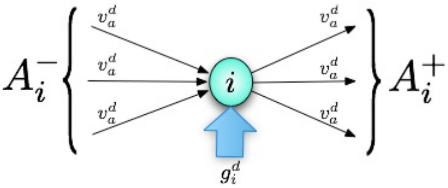
MTE equations

Expected in-flow

$$x_i^d = g_i^d + \sum_{a \in A_i^-} v_a^d$$

leaves node i according to

$$v_a^d = x_i^d \, \mathbb{P}(\tilde{t}_a + \tilde{\tau}_{j_a}^d \leq \tilde{t}_b + \tilde{\tau}_{j_b}^d \ \forall\, b \in A_i^+)$$



with $t_a = s_a(w_a)$ and $w_a = \sum_d v_a^d$

Variational formulation

$$\tilde{\tau}_i^d = \min_{a \in A_i^+} \{\tilde{t}_a + \tilde{\tau}_{j_a}^d\}$$

Theorem (Baillon-C'06)

$\tau_i^d = \mathbb{E}(\tilde{\tau}_i^d)$ is the unique solution of the stochastic Bellman equations

$$\begin{cases} \tau_d^d = 0 \\ \tau_i^d = \mathbb{E}(\min_{a \in A_i^+} \{t_a + \tau_{j_a}^d + \varepsilon_a^d\}) \end{cases}$$

Moreover $t \mapsto \tau_i^d(t)$ is concave & smooth.

Variational formulation

Theorem (Baillon-C'06)

MTE is characterized by

$$(D) \qquad \operatorname*{Min}_{t} \ \phi(t) \triangleq \sum_{a} \int_0^{t_a} s_a^{-1}(x)\, dx - \sum_{i,d} g_i^d \tau_i^d(t)$$

...same form as Wardrop equilibrium!

Method of Successive Averages

Algorithm 2 MSA - main iteration

1: Compute current arc travel times $t_a^n = s_a(w_a^n)$
2: Solve stochastic Bellman's equations
3: Compute invariant measures of Markov chains $\tilde{v}_a^d$
4: Aggregate flows $\tilde{w}_a^n = \sum \tilde{v}_a^d$
5: Update $w^{n+1} = (1-\alpha_n)w^n + \alpha_n \tilde{w}^n$

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$$\frac{w^{n+1}-w^n}{\alpha_n} = -\nabla\phi(t^n)$$

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$$\frac{w^{n+1}-w^n}{\alpha_n} = -\nabla\phi(t^n) = -D(w^n)^{-1}\nabla\tilde{\phi}(w^n)$$

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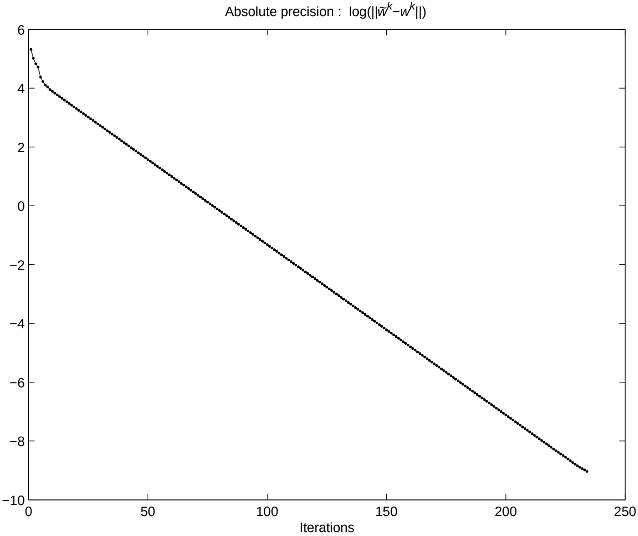
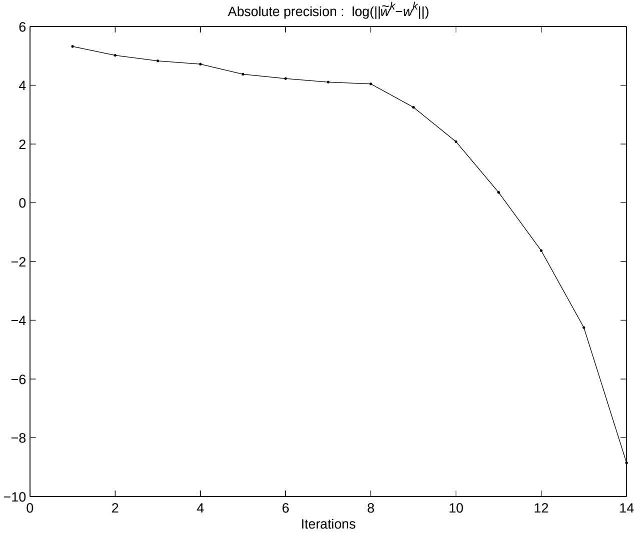
Theorem (Baillon-C'06)

$\sum \alpha_n = \infty$ *and* $\sum \alpha_n^2 < \infty \Rightarrow$ *convergence to MTE*

Stochastic MSA iterations



Stochastic MSA-Newton iterations



Atomic equilibrium in congestion games

- A finite set of players $i \in I$ traveling from o_i to d_i
- Each player i selects a path $r_i \in \mathcal{R}_i$
- These choices induce arc loads $u_a = \#\{i : a \in r_i\}$
- Player i experiences a travel time $c_i(r_i, r_{-i}) = \sum_{a \in r_i} s_a(u_a)$

Definition

A *pure Nash equilibrium* is a strategy profile $(r_i)_{i \in I}$ so that for each i

$$c_i(r_i, r_{-i}) \leq c_i(r_i', r_{-i}) \qquad \forall\ r_i' \in \mathcal{R}_i$$

Example: 50%-50% split between 2 identical routes

Mixed equilibrium

- Mixed strategies $\pi^i = (\pi^{ir})_{r\in\mathcal{R}_i} \in \Delta(\mathcal{R}_i)$
- Expected costs

$$c_i(\pi^i, \pi^{-i}) = \mathbb{E}_\pi(c_i(r_i, r_{-i})) = \sum_{r\in\mathcal{R}_i} \pi^{ir} \sum_{a\in r} \mathbb{E}(s_a(1+u_a^{-i})).$$

where $u_a^{-i} = \#\{j \neq i : a \in r_j\}$.

Definition

A *mixed Nash equilibrium* is a strategy profile $(\pi^i)_{i\in I}$ so that for all i

$$c_i(\pi^i, \pi^{-i}) \leq c_i(r, \pi^{-i}) \qquad \forall\ r \in \Delta(\mathcal{R}_i)$$

Multiple mixed equilibria... Examples with 2 identical routes

Rosenthal's potential

Theorem (Rosenthal'73)

Consider the potential function

$$\Phi((r_i)_{i\in I}) = \sum_{a\in A}\sum_{j=1}^{u_a} s_a(j).$$

Then for each player $i \in I$ and every alternative path $r_i' \neq r_i$

$$\Phi(r_i', r_{\text{-}i}) - \Phi(r_i, r_{\text{-}i}) = c_i(r_i', r_{\text{-}i}) - c_i(r_i, r_{\text{-}i}).$$

Corollary

a) *There exist pure Nash equilibria: any (local) minimum of $\Phi(\cdot)$*

b) *Best response dynamics converge in finitely many iterations to a Nash equilibrium in pure strategies... but require full information !*

Rosenthal's potential – Proof

If player i changes from r_i to r_i' the new loads are

$$u_a' = \begin{cases} u_a+1 & \text{for } a \in r_i' \setminus r_i \\ u_a-1 & \text{for } a \in r_i \setminus r_i' \\ u_a & \text{otherwise} \end{cases}$$

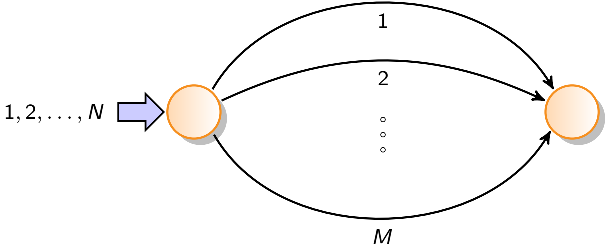
$$\begin{aligned} \Phi(r_i', r_{-i}) - \Phi(r_i, r_{-i}) &= \sum_{a \in r_i' \setminus r_i} s_a(u_a+1) - \sum_{a \in r_i \setminus r_i'} s_a(u_a) \\ &= \sum_{a \in r_i'} s_a(u_a') - \sum_{a \in r_i} s_a(u_a) \\ &= c_i(r_i', r_{-i}) - c_i(r_i, r_{-i}) \end{aligned}$$

□

Adaptive dynamics and equilibrium

Dynamical models that sustain equilibrium? (C-Melo-Sorin'10)

$$i = 1, \ldots, N \quad \text{drivers}$$
$$r = 1, \ldots, M \quad \text{routes}$$



c_u^r = travel time of route r under a load of u drivers

Adpative dynamics in repeated games

Fictitious play, stochastic fictitious play, reinforcement dynamics, replicator dynamics, asymptotic calibration... dozens of papers in last 20 years

Fudenberg D., Levine D.K., *The Theory of Learning in Games*
MIT Press (1998)

Hofbauer J., Sigmund K., *Evolutionary Games and Population Dynamics*
Cambridge University Press (1998)

Young P., *Strategic Learning and its Limits*
Oxford University Press (2004)

Sandholm W., *Population Games and Evolutionary Dynamics*
Forthcoming (2011)

Discrete stochastic adaptive learning process

State variable: x^{ir} = perception of driver i on route r

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Dynamics:

$$\underset{\text{state}}{x^{ir}_{n-1}} \rightsquigarrow \underset{\text{pbb's}}{\pi^{ir}_n} \rightsquigarrow \underset{\text{routes}}{Y^{ir}_n} \rightsquigarrow \underset{\text{loads}}{u^r_n} \rightsquigarrow \underset{\text{costs}}{c^r_{u^r_n}} \rightsquigarrow \underset{\text{update}}{x^{ir}_n}$$

Minimal information: Players only observe their own payoff !

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Dynamics:

$$x_n^{ir} = \begin{cases} (1-\alpha_n)x_{n-1}^{ir} + \alpha_n c^r_{u^r_n} & \text{if } Y_n^{ir} = 1 \\ x_{n-1}^{ir} & \text{if } Y_n^{ir} = 0 \end{cases}$$

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Stochastic Approximation: basic framework
(Robbins-Monro'51, Ljung'71,..., Benaim-Hirsch'96)

A Robbins-Monro process is a stochastic process of the form

$$(RM) \qquad \boxed{\frac{x_{n+1}-x_n}{\alpha_{n+1}} = F(x_n) + u_{n+1}}$$

with u_n a sequence of random variables adapted to a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$: u_n is $\mathcal{F}_n$-measurable with $\mathbb{E}(u_{n+1}|\mathcal{F}_n) = 0$.

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Such a process can be interpreted as a stochastically perturbed discretization of the differential equation

$$(DD) \qquad \boxed{\frac{dx}{dt} = F(x)}$$

Stochastic Approximation: attractors and convergence

Under the following conditions (with $p \geq 2$)

- x_n bounded
- u_n bounded in L^p
- $\sum \alpha_n = \infty$ and $\sum \alpha_n^{1+p/2} < \infty$

the ω-limit set of the sequence $(x_n)_{n\in\mathbb{N}}$ generated by (RM) is $\mathbb{P}$-almost surely a compact set which is invariant for (DD) with no proper attractor.

insert figure ICT

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insert figure ICT

Theorem

Under the assumptions above

1. *If x^* is a global attractor of (DD) then $\mathbb{P}(x_n \to x^*) = 1$*
2. *If x^* is a local attractor of (DD) then $\mathbb{P}(x_n \to x^*) > 0$*

Stochastic Approximation: example statistical estimation

(Robbins-Monro'51)

Problem: Estimate the intensity $x \geq 0$ for a radiation therapy which allows to reduce a tumor by a fraction ρ (in expected value).

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Treatment effectivity is a ***bounded*** random variable $Y \sim \mathcal{F}(x)$ with $\mathbb{E}(Y) = M(x)$ an unknown increasing function of x. We assume that there is a unique solution θ of the equation $M(\theta) = \rho$.

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We observe outcomes $y_n = Y(x_n)$ at levels $x_0, x_1, x_2, \ldots$ and update

$$x_{n+1} = x_n + \alpha_{n+1}(\rho - y_n).$$

with $(\alpha_n)_{n \in \mathbb{N}} \in \ell^2 \setminus \ell^1$.

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with $(\alpha_n)_{n\in\mathbb{N}} \in \ell^2 \setminus \ell^1$. The corresponding ODE

$$\tfrac{dx}{dt} = \rho - M(x)$$

has θ as its unique global attractor so that $x_n \to \theta$ almost surely.

Stochastic Approximation: example law of large numbers

Let $(Y_k)_{k\in\mathbb{N}}$ be a sequence of i.i.d. *bounded* random variables with expected value μ. Let $x_n = \frac{1}{n}(Y_1 + \cdots + Y_n)$.

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The corresponding ODE is

$$\frac{dx}{dt} = \mu - x$$

whose solution is exponential with $x(t) \to \mu$, thus $x_n \to \mu$ almost surely.

Discrete stochastic adaptive learning process

Back to adaptive learning in the atomic congestion game

State variable: $x^{ir} \quad = \quad$ perception of driver i on route r

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Continuous-time adaptive dynamics

(LP) $$\boxed{\frac{x_n - x_{n-1}}{\alpha_n} = \tilde{V}_n}$$ Learning process

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Mean field approximation: if $\sum \alpha_n = \infty$ and $\sum \alpha_n^2 < \infty$

(AD) $$\boxed{\frac{dx}{dt} = \mathbb{E}(\tilde{V}|x)}$$ Adaptive dynamics

Analytic expression for the mean field

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$$\overbrace{\sum_{u=1}^{N-1} c^r_{1+u} \sum_{|A|=u} \prod_{j\in A} \pi^{jr} \prod_{j\notin A}(1-\pi^{jr})}$$

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Adaptive dynamics

$$\boxed{\frac{dx^{ir}}{dt} = \pi^{ir}(x)[C^{ir}(x) - x^{ir}]}$$

$$\begin{aligned} C^{ir}(x) &= F^{ir}(\Pi(x)) \\ \Pi(x) &= (\pi^{ir}(x)) \end{aligned}$$

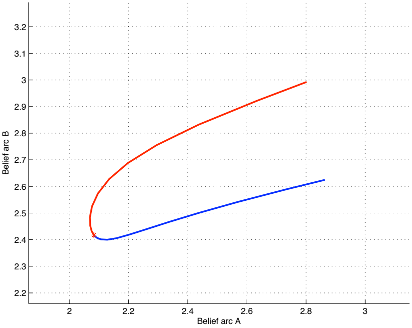
Simulation: 2 drivers × 2 routes

$$\frac{dx}{dt}^{1a} = \pi^a(x^1)[C^a(x^2) - x^{1a}] \qquad \text{(driver 1)}$$
$$\frac{dx}{dt}^{1b} = \pi^b(x^1)[C^b(x^2) - x^{1b}]$$

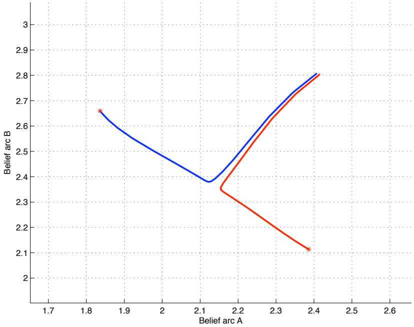
$$\frac{dx}{dt}^{2a} = \pi^a(x^2)[C^a(x^1) - x^{2a}] \qquad \text{(driver 2)}$$
$$\frac{dx}{dt}^{2b} = \pi^b(x^2)[C^b(x^1) - x^{2b}]$$

$$\pi^a(x) = \exp(-\beta x^a)/[\exp(-\beta x^a) + \exp(-\beta x^b)]$$
$$\pi^b(x) = \exp(-\beta x^b)/[\exp(-\beta x^a) + \exp(-\beta x^b)]$$
$$C^a(x) = c_1^a \pi^b(x) + c_2^a \pi^a(x)$$
$$C^b(x) = c_1^b \pi^a(x) + c_2^b \pi^b(x)$$

Simulation: 2 drivers $\times$ 2 routes

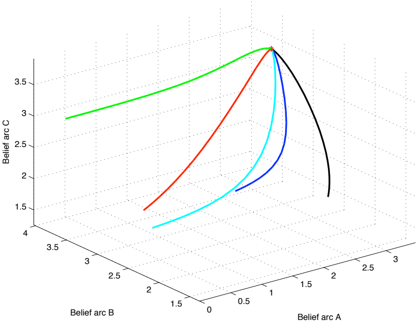


$\beta = 1.0$

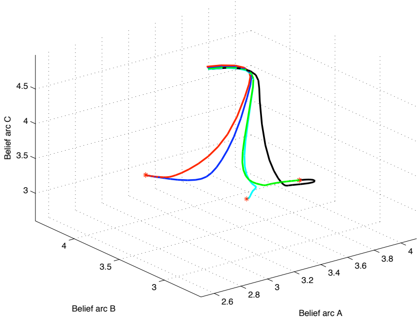


$\beta = 2.5$

Simulation: 5 drivers $\times$ 3 routes

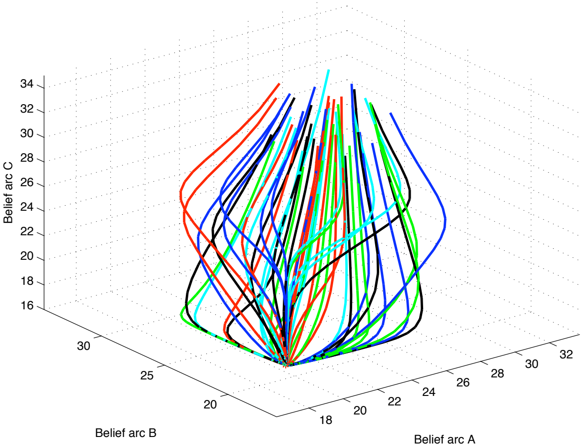


$\beta = 1.0$



$\beta = 3.0$

Simulation: 50 drivers $\times$ 3 routes



$\beta = 0.3$

Rest points — an underlying game

$\mathcal{E} = \{\text{rest points}\} = \{x : x^{ir} = C^{ir}(x) \text{ for all } i, r\}$

$$x = C(x) = T(\Pi(x)) \Leftrightarrow \begin{cases} x = T(\pi) \\ \pi = \Pi(x) \end{cases}$$

Thus $x \rightleftarrows \pi$ bijects $\mathcal{E}$ with $\Pi(\mathcal{E}) = \{\text{rest probabilities}\}$

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Theorem (C-Melo-Sorin'10)

$\Pi(\mathcal{E}) =$ *Nash equilibria of the N-person game with strategies* $\pi^i \in \Delta(R)$ *and costs*

$$G^i(\pi) = \langle \pi^i, F^i(\pi) \rangle + \tfrac{1}{\beta} \sum_r \pi^{ir}[\ln \pi^{ir} - 1]$$

Rest points — existence/uniqueness/convergence

Denote $\delta = \max_{r,u}[c_u^r - c_{u-1}^r]$ the maximal congestion jump

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2. *Exactly one of them is symmetric:* $\hat{x}^{ir} = \hat{x}^{jr}$

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3. $\beta\delta < 2 \quad \Rightarrow \quad \hat{x}$ *is the unique rest point and a local attractor*
4. $\beta\delta < \frac{2}{N-1} \quad \Rightarrow \quad \hat{x}$ *is a global attractor* $\quad \Rightarrow \quad \mathbb{P}(x_n \to \hat{x}) = 1$

Potential function

Theorem (C-Melo-Sorin'10)

The map F admits a potential, namely $F(\pi) = \nabla H(\pi)$ where

$$H(\pi) = \sum_r \mathbb{E}(c_1^r + c_2^r + \cdots + c_{U^r}^r).$$

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Denote

$$\begin{aligned} H_\beta(\pi) &= H(\pi) + \tfrac{1}{\beta} \textstyle\sum_{ir} \pi^{ir} \ln(\pi^{ir}) \\ \mathcal{L}(\pi;\lambda) &= H_\beta(\pi) - \textstyle\sum_i \lambda^i [\sum_r \pi^{ir} - 1] \end{aligned}$$

Equivalent Lagrangian dynamics

The adaptive dynamics can be written

$$\frac{dx}{dt} = -\tfrac{1}{\beta}\nabla_x L(x;\lambda(x))$$

where

$$\begin{aligned} L(x;\lambda) &= \mathcal{L}(\pi(x,\lambda);\lambda) \\ \pi^{ir}(x,\lambda) &= \exp(-\beta(x^{ir}-\lambda^i)) \\ \lambda^i(x) &= -\tfrac{1}{\beta}\ln(\textstyle\sum_r \exp(-\beta x^{ir})) \end{aligned}$$

Rest points as extremals

Theorem (C-Melo-Sorin'10)

For $\pi = \Pi(x)$ the following are equivalent

(a) $x \in \mathcal{E}$

(b) $\nabla_x L(x, \lambda(x)) = 0$

(c) π *is a Nash equilibrium*

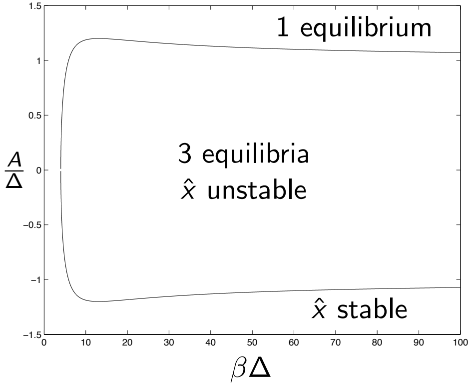
(d) $\nabla_\pi \mathcal{L}(\pi, \lambda) = 0$ *for some* $\lambda \in \mathbb{R}^M$

(e) π *is a critical point of* $H_\beta(\cdot)$ *on* $\Delta(R)^N$

Moreover, if $\beta\delta < 1$ *then* $H_\beta(\cdot)$ *is strongly convex and* $\hat{\pi} = \Pi(\hat{x})$ *is its unique minimizer on* $\Delta(R)^N$.

Rest points — Bifurcation: 2 drivers $\times$ 2 routes

Symmetric equilibrium $\hat{x}$ is stable $\Leftrightarrow$ $|\frac{A}{\Delta}| > h(\frac{4}{\beta\Delta})$

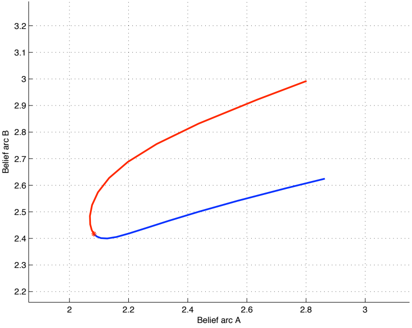


$$h(z) = \sqrt{1-z} + z \tanh^{-1}\sqrt{1-z}$$

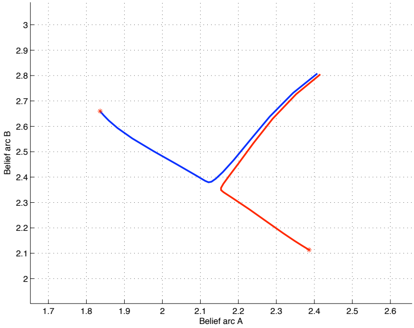
$$A = (c_2^a + c_1^a) - (c_2^b + c_1^b)$$

$$\Delta = (c_2^a - c_1^a) + (c_2^b - c_1^b)$$

Bifurcation: 2 drivers $\times$ 2 routes



$\beta = 1.0$



$\beta = 2.5$

State dependent update — Mario Bravo 2012

Players exploit memory of play for updating

$$x_n^{ir} - x_{n-1}^{ir} = \frac{1}{\theta_n^{ir}} Y_n^{ir} [c_{u_n^r}^r - x_n^{ir}]$$

with θ_n^{ir} the number of times route r has been used by i up to time n.

State dependent update — Mario Bravo 2012

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with θ_n^{ir} the number of times route r has been used by i up to time n.

The empirical frequencies of play $\pi_n^{ir} = \theta_n^{ir}/n$ satisfy the recursion

$$\pi_n^{ir} - \pi_{n-1}^{ir} = \tfrac{1}{n}\left(\mathbb{1}_{\{r_n^i=r\}} - \pi_{n-1}^{ir}\right)$$

State dependent update — Mario Bravo 2012

MB's process leads to the coupled adaptive dynamics

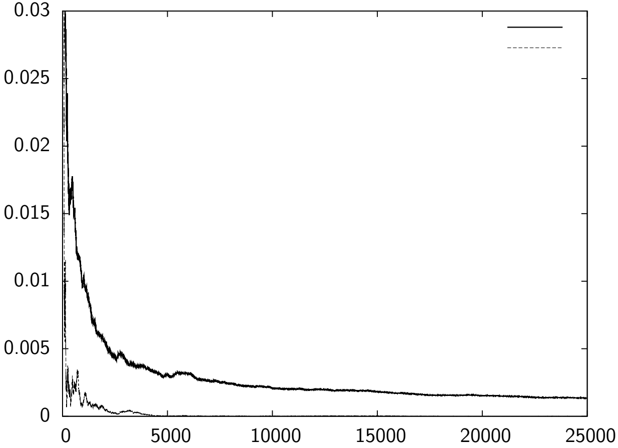
(CAD)
$$\begin{cases} \dot{x}^{ir} = \frac{\pi^{ir}(x)}{\pi^{ir}}[C^{ir}(x) - x^{ir}] \\ \dot{\pi}^{ir} = \pi^{ir}(x) - \pi^{ir} \end{cases}$$

Theorem (Bravo'12)

1. *Same rest points:* $x^* \in \mathcal{E}$, $\pi^* = \pi(x^*)$
2. $\beta\delta < 2 \quad \Rightarrow \quad$ *convergence with positive probability*
3. $\beta\delta < \frac{2}{N-1} \quad \Rightarrow \quad$ *almost sure convergence*

Comparison of discrete dynamics speeds

$$\|(x_n, \pi_n) - (x^*, \pi^*)\| \quad \text{vs} \quad \|x_n - x^*\|$$



Extensions and open problems

- Extended to finite games and general discrete choice models
- Applications to multipath TCP/IP protocol design

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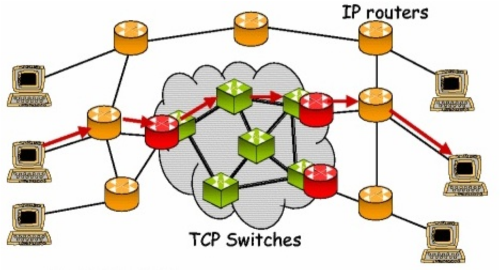
- Applications to multipath TCP/IP protocol design

- Open problems
  - Almost sure convergence beyond bifurcation threshold?
  - Speed of convergence and large deviations?
  - Understand general structure of rest point bifurcation?
  - More realistic adaptive learning dynamics?
  - Connections with classical equilibrium models?

Internet traffic control — TCP/IP

TCP/IP – Single path routing

- $G = (N, A)$ communication network
- Each source $s \in S$ transmits packets from origin o_s to destination d_s
- Along which route? At which rate?



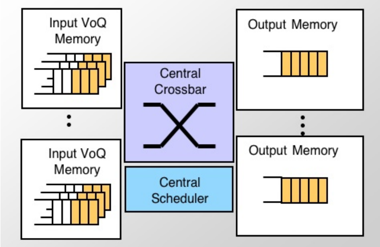
TCP/IP – Current protocols

- **Route selection** (RIP/OSPF/IGRP/BGP/EGP)
  Dynamic adjustment of routing tables
  Slow timescale evolution (15-30 seconds)
  Network Layer 3

- **Rate control** (TCP Reno/Tahoe/Vegas)
  Dynamic adjustment of source rates – congestion window
  Fast timescale evolution (100-300 milliseconds)
  Transport Layer 4

Congestion measures: link delays / packet loss

Switch/Router



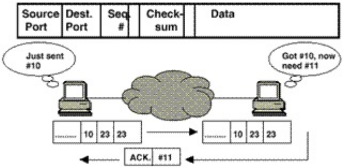
- Links have random delays $\tilde{\lambda}_a = \lambda_a + \epsilon_a$ with $\mathbb{E}(\epsilon_a) = 0$

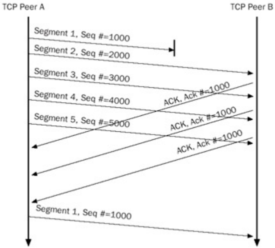
$$\tilde{\lambda}_a = \text{queuing} + \text{transmission} + \text{propagation}$$

- And packet loss probabilities p_a because of finite queuing buffers

TCP – Congestion window

Packets $\longleftrightarrow$ Acks





$$x_s = \text{source rate} \sim \frac{\text{congestion window}}{\text{round-trip time}} = \frac{W_s}{\tau_s}$$

TCP – Congestion control

Sources adjust transmission rates in response to congestion

Basic principle: higher congestion $\Leftrightarrow$ *smaller rates*

λ_a : link congestion measure (loss pbb, queuing delay)

x_s : source transmission rate [packets/sec]

$q_s = \sum_{a \in s} \lambda_a$ (end-to-end congestion)

$y_a = \sum_{s \ni a} x_s$ (aggregate link loads)

Decentralized algorithms

$$
\begin{aligned}
x_s^{t+1} &= F_s(x_s^t, q_s^t) \quad \text{(TCP – source dynamics)} \\
\lambda_a^{t+1} &= G_a(\lambda_a^t, y_a^t) \quad \text{(AQM – link dynamics)}
\end{aligned}
$$

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Example: TCP-Reno / packet loss probability

AIMD control

$$W_s^{t+\tau_s} = \begin{cases} W_s^t + 1 & \text{if } W_s^t \text{ packets are successfully transmitted} \\ \lceil W_s^t/2 \rceil & \text{one or more packets are lost (duplicate ack's)} \end{cases}$$

$$\pi_s^t = \prod_{a \in s}(1-p_a^t) = \text{success probability (per packet)}$$

Additive congestion measure

$$\left.\begin{array}{l} q_s^t \triangleq -\ln(\pi_s^t) \\ \lambda_a^t \triangleq -\ln(1-p_a^t) \end{array}\right\} \Rightarrow q_s^t = \sum_{a \in s} \lambda_a^t$$

Approximate model for rate dynamics

$$\mathbb{E}(W_s^{t+\tau_s} | W_s^t) \sim e^{-q_s^t W_s^t}(W_s^t + 1) + (1 - e^{-q_s^t W_s^t})\lceil W_s^t/2 \rceil$$

$$\Rightarrow \boxed{x_s^{t+1} = x_s^t + \frac{1}{2\tau_s}\left[e^{-\tau_s q_s^t x_s^t}(x_s^t + \tfrac{2}{\tau_s}) - x_s^t\right]}$$

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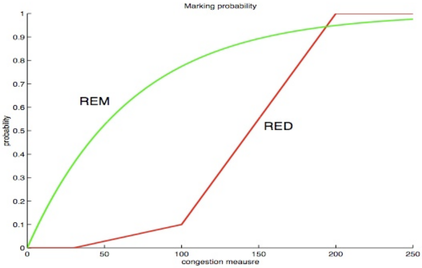
Example: AQM / Droptail $\longrightarrow$ RED-REM

Marking probability on links controlled by AQM

$$p_a^t = \varphi_a(r_a^t)$$

as a function of average queue length

$$r_a^{t+1} = (1-\alpha) r_a^t + \alpha L_a^t$$



Loss probability vs. average queue length

Network Utility Maximization

- Kelly, Maullo and Tan (1998) proposed an optimization-based model for distributed rate control in networks.
- Low, Srikant, etc. (1999-2002) showed that current TCP/AQM control algorithms solve an implicit network optimization problem.
- During last decade, the model has been used and extended to study the performance of wired and wireless networks.

Steady state equations

$$
\begin{aligned}
x_s^{t+1} &= F_s(x_s^t, q_s^t) && \text{(TCP – source dynamics)} \\
\lambda_a^{t+1} &= G_a(\lambda_a^t, y_a^t) && \text{(AQM – link dynamics)}
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$$\Updownarrow$$

$$\boxed{\begin{array}{ll} x_s = f_s(q_s) & \text{(decreasing)} \\ \lambda_a = \psi_a(y_a) & \text{(increasing)} \\ q_s = \sum_{a \in s} \lambda_a & \\ y_a = \sum_{s \ni a} x_s & \end{array}}$$

Steady state equations

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x_s & = & F_s(x_s, q_s) & \text{(TCP – source equilibrium)} \\
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\end{array}}
\Leftrightarrow
\boxed{\begin{array}{l}
x_s = f_s(\sum_{a \in s} \lambda_a) \\
\lambda_a = \psi_a(\sum_{s \ni a} x_s)
\end{array}}
$$

Examples

TCP-Reno (loss probability)

$$q_s = f_s^{-1}(x_s) \triangleq \frac{1}{\tau_s x_s} \ln(1 + \frac{2}{\tau_s x_s})$$

$$\lambda_a = \psi_a(y_a) \triangleq \frac{\delta y_a}{c_a - y_a}$$

TCP-Vegas (queueing delay)

$$q_s = f_s^{-1}(x_s) \triangleq \frac{\alpha \tau_s}{x_s}$$

$$\lambda_a = \psi_a(y_a) \triangleq \frac{y_a}{c_a - y_a}$$

Steady state – Primal optimality

$$\boxed{\begin{aligned} x_s &= f_s(\textstyle\sum_{a\in s}\lambda_a) \\ \lambda_a &= \psi_a(\textstyle\sum_{s\ni a} x_s) \end{aligned}}$$

$$f_s^{-1}(x_s) = \textstyle\sum_{a\in s}\lambda_a = \sum_{a\in s}\psi_a(\sum_{u\ni a} x_u)$$

$\equiv$ optimal solution of strictly convex program

$$(P) \quad \boxed{\min_x \ \textstyle\sum_{s\in S} U_s(x_s) + \sum_{a\in A}\Psi_a(\sum_{s\ni a} x_s)}$$

$$\begin{aligned} U_s'(\cdot) &= -f_s^{-1}(\cdot) \\ \Psi_a'(\cdot) &= \psi_a(\cdot) \end{aligned}$$

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$$\psi_a^{-1}(\lambda_a) = \textstyle\sum_{s\ni a} x_s = \sum_{s\ni a} f_s(\sum_{b\in s}\lambda_b)$$

$\equiv$ optimal solution of strictly convex program

$$(D) \quad \boxed{\min_{\lambda} \ \textstyle\sum_{a\in A}\Psi_a^*(\lambda_a) + \sum_{s\in S} U_s^*(\sum_{a\in s}\lambda_a)}$$

Steady state – Dual optimality

$$\boxed{\begin{aligned} x_s &= f_s\left(\textstyle\sum_{a\in s}\lambda_a\right)\\ \lambda_a &= \psi_a\left(\textstyle\sum_{s\ni a}x_s\right)\end{aligned}}$$

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$$(D)\qquad \boxed{\min_{\lambda}\ \textstyle\sum_{a\in A}\Psi_a^*(\lambda_a)+\sum_{s\in S}U_s^*\left(\sum_{a\in s}\lambda_a\right)}$$

Theorem (Low'2003)

$$\boxed{\begin{array}{l} x_s = f_s(\sum_{a\in s} \lambda_a) \\ \lambda_a = \psi_a(\sum_{s\ni a} x_s) \end{array}} \Leftrightarrow \boxed{\begin{array}{l} x \text{ and } \lambda \text{ are optimal solutions} \\ \text{for } (P) \text{ and } (D) \text{ respectively} \end{array}}$$

Relevance:

- Reverse engineering of existing protocols / forward engineering (f_s, ψ_a)
- Design distributed stable protocols to optimize prescribed utilities
- Flexible choice of congestion measure q_s

Limitations:

- Ignores delays in transmission of congestion signals
- Improper account of stochastic phenomena
- Single-path routing

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- Reverse engineering of existing protocols / forward engineering (f_s, ψ_a)
- Design distributed stable protocols to optimize prescribed utilities
- Flexible choice of congestion measure q_s

Limitations:

- Ignores delays in transmission of congestion signals
- Improper account of stochastic phenomena
- Single-path routing

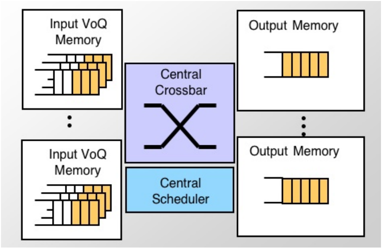
Markovian Network Utility Maximization (MNUM)

- Increase transmission rates: single path $\longrightarrow$ multi-path
- Goal: design distributed TCP protocols with multi-path routing
- Packet-level protocol that is stable and satisfies optimality criteria
- Model based on the notion of Markovian traffic equilibrium

MNUM: integrated routing & rate control

- Cross-layer design: routing + rate control
- Based on a common congestion measure: delay
- Link random delays $\tilde{\lambda}_a = \lambda_a + \epsilon_a$ with $\mathbb{E}(\epsilon_a) = 0$

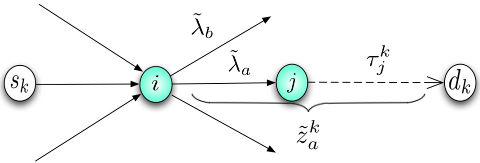
$$\tilde{\lambda}_a = \text{queuing} + \text{transmission} + \text{propagation}$$



MNUM: Markovian multipath routing

At switch i, packets headed to destination d are routed through the outgoing link $a \in A_i^+$ that minimizes the *"observed"* cost-to-go

$$\tilde{\tau}_i^d = \min_{a \in A_i^+} \underbrace{\tilde{\lambda}_a + \tau_{j_a}^d}_{\tilde{z}_a^d}$$



Markov chain with transition matrix

$$P_{ij}^d = \begin{cases} \mathbb{P}(\tilde{z}_a^d \leq \tilde{z}_b^d, \forall b \in A_i^+) & \text{if } i = i_a, j = j_a \\ 0 & \text{otherwise} \end{cases}$$

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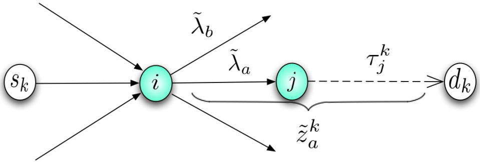

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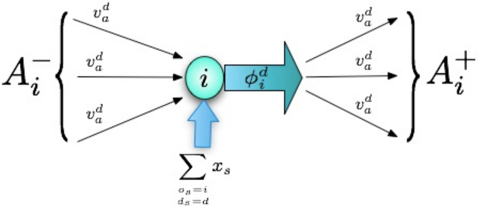
Expected flows (invariant measures)

The flow ϕ_i^d entering node i and directed towards d

$$\phi_i^d = \sum_{\substack{o_s=i \\ d_s=d}} x_s + \sum_{a\in A_i^-} v_a^d$$

splits among the outgoing links $a = (i, j)$ according to

$$v_a^d = \phi_i^d P_{ij}^d$$



Expected costs

Letting $z_a^d = \mathbb{E}(\tilde{z}_a^d)$ and $\tau_i^d = \mathbb{E}(\tilde{\tau}_i^d)$, we have

$$z_a^d = \lambda_a + \tau_{j_a}^d$$
$$\tau_i^d = \varphi_i^d(z^d)$$

with

$$\boxed{\varphi_i^d(z^d) \triangleq \mathbb{E}(\min_{a \in A_i^+}[z_a^d + \epsilon_a^d])}$$

Moreover

$$\mathbb{P}\left(\tilde{z}_a^d \leq \tilde{z}_b^d, \forall b \in A_i^+\right) = \frac{\partial \varphi_i^d}{\partial z_a^d}(z^d)$$

Expected costs

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Markovian NUM – Definition

$$
\begin{array}{ll}
x_s = f_s(q_s) & \text{(source rate control)} \\
\lambda_a = \psi_a(y_a) & \text{(link congestion)} \\
y_a = \sum_d v_a^d & \text{(total link flows)} \\
q_s = \tau_s - \tau_s^0 & \text{(end-to-end queuing delay)}
\end{array}
$$

where $\tau_s = \tau_{o_s}^{d_s}$ with expected costs given by

$$
(ZQ) \quad \left\{ \begin{array}{l} z_a^d = \lambda_a + \tau_{j_a}^d \\ \tau_i^d = \varphi_i^d(z^d) \end{array} \right.
$$

and expected flows v^d satisfying

$$
(FC) \quad \left\{ \begin{array}{ll} \phi_i^d = \sum_{\substack{o_s=i \\ d_s=d}} x_s + \sum_{a \in A_i^-} v_a^d & \forall i \neq d \\ v_a^d = \phi_i^d \frac{\partial \varphi_i^d}{\partial z_a^d}(z^d) & \forall a \in A_i^+ \end{array} \right.
$$

MNUM Characterization: Dual problem

- (ZQ) defines implicitly z_a^d and τ_i^d as concave functions of λ
- $x_s = f_s(q_s)$ with $q_s = \tau_{o_s}^{d_s}(\lambda) - \tau_{o_s}^{d_s}(\lambda^0)$ yields x_s as a function of λ
- (FC) then defines v_a^d as functions of λ

$$\text{MNUM conditions} \quad \Leftrightarrow \quad \psi_a^{-1}(\lambda_a) = y_a = \textstyle\sum_d v_a^d(\lambda)$$

Theorem

MNUM $\Leftrightarrow$ optimal solution of the strictly convex program

$$(D) \quad \min_{\lambda} \sum_{a \in A} \Psi_a^*(\lambda_a) + \sum_{s \in S} U_s^*(q_s(\lambda))$$

MNUM Characterization: Primal problem

Theorem

MNUM $\Leftrightarrow$ optimal solution of

$$\min_{(x,y,v)\in P} \sum_{s\in S} U_s(x_s) + \sum_{a\in A} \Psi_a(y_a) + \sum_{d\in D} \chi^d(v^d)$$

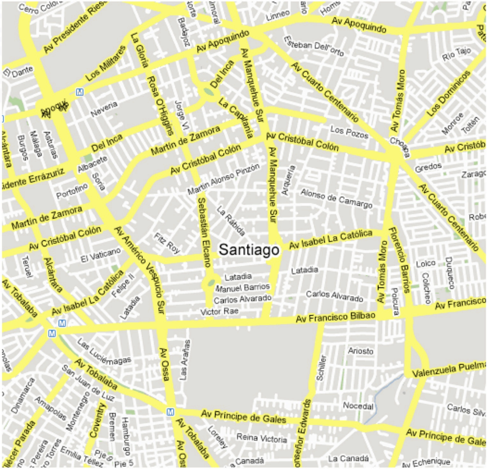
where

$$\chi^d(v^d) = \sup_{z^d} \sum_{a\in A} (\varphi^d_{i_a}(z^d) - z^d_a) v^d_a$$

and P is the polyhedron defined by flow conservation constraints.

Risk averse routing

What is the risk of a path?



Copenhagen – DTU Transport (www.transport.dtu.dk)

Figure 2: Example of real time illustration of congestion (Source: Vejdirektoratet, www.trafikken.dk)

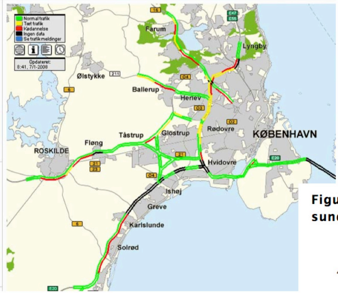
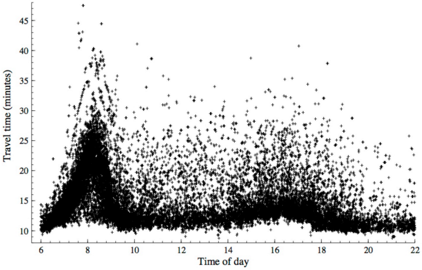
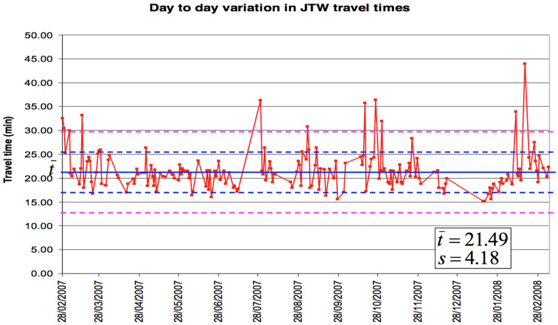


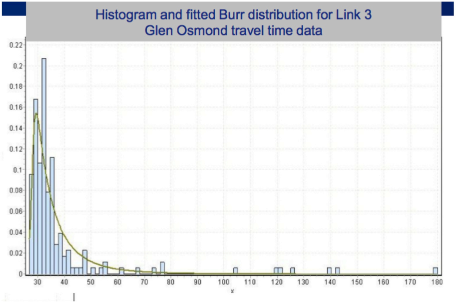
Figure 7: Observations of travel time by time of day. Frederikssundsvej, inward direction

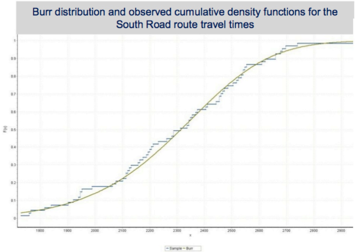


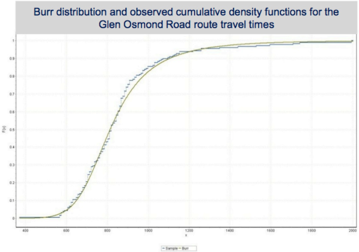
Adelaide, South Australia (Susilawati *et al.* 2011)



Previous: Normal, Log-normal, Gamma, Weibull
Best fit: Burr distribution $F(x) = 1 - (1 + x^c)^{-k}$







Some recent literature on risk averse routing

[1] Loui – *Optimal paths in graphs with stochastic or multidimensional weights.* Commun. ACM 26(9), 1983.

[2] Bates *et al.* – *The evaluation of reliability for personal travel.* Transportation Research E 37, 2001.

[3] Noland, Polak – *Travel time variability: a review of theoretical and empirical issues.* Transport Reviews 22, 2002.

[4] Hollander – *Direct versus indirect models for the effects of unreliability.* Transportation Research A 40, 2006.

[5] Nie, Wu – *Shortest path problem considering on-time arrival probability*. Transportation Research A 40, 2006.

[6] Ordóñez & Stier-Moses – *Wardrop equilibria with risk-averse users.* Transportation Science 44(1), 2010.

[7] Engelson & Fosgerau – *Additive measures of travel time variability.* Transportation Research B 45, 2011.

Some recent literature on risk averse routing

[8] Nie – *Multiclass percentile user equilibrium with flow dependent stochasticity*. Transportation Research B 45(10), 2011.

[9] Wu, Nie – *Modeling heterogeneous risk-taking behavior in route choice*. Transportation Research A 45(9), 2011.

[10] Nie, Wu, Homem-de-Mello – *Optimal path problems with second-order stochastic dominance constraints*. Networks & Spatial Economics 12(4), 2012.

[11] Nikolova & Stier-Moses – *A mean-risk model for the traffic assignment problem with stochastic travel times*. Operations Research 62(2), 2014.

[12] Jaillet, Qi & Sim – *Routing optimization with deadlines under uncertainty*. To appear in Operations Research.

[13] Cominetti, Torrico – *Additive consistency of risk measures and its application to risk-averse routing in networks*. To apear in Mathematics of Operations Research.

In this session. . .

1. How do we measure the risk of a path?
2. Some risk measures — paradoxes and drawbacks
3. Additive consistency — entropic risk measures
4. Remarks — optimal paths and network equilibrium
5. Remarks — dynamic risk measures

Setting

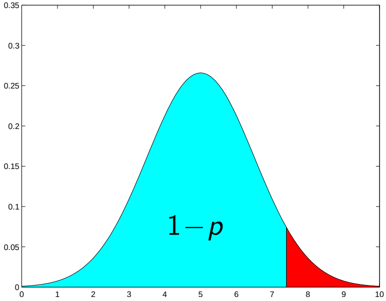
- Bounded random variables: $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$
- Preferences: $X \preceq Y \Leftrightarrow \Phi(X) \leq \Phi(Y)$
- Scalar measure of risk: $\Phi(X) \in \mathbb{R}$

Some popular risk measures

$$\phi(X) = \mu_X + \gamma\sigma_X \qquad \text{(Markowitz)}$$

$$\phi(X) = VaR_p(X) = (1-p)\text{-percentile} \qquad \text{(Value-at-Risk)}$$

$$\phi(X) = AVaR_p(X) = \mathbb{E}[X | X \geq VaR_p(X)] \qquad \text{(Average VaR)}$$



Two natural axioms

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if $X \leq Y$ almost surely then $\phi(X) \leq \phi(Y)$

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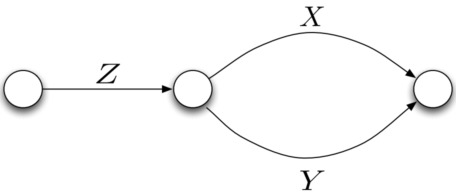

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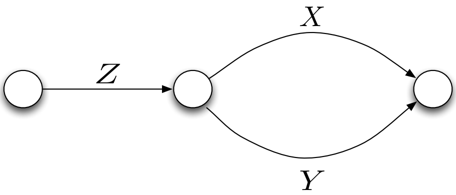
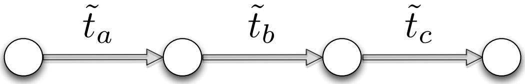


Additive consistency fails for Markowitz, VaR, CVaR.
Markowitz not even monotone.

How to measure risk: mean-stdev (Markowitz 1952)



$$X = \sum_{a\in r} \tilde{t}_a$$

$$\Phi_\gamma(X) = \mu + \gamma\sigma = \sum_{a\in r} \mu_a + \gamma\sqrt{\sum_{a\in r} \sigma_a^2}$$

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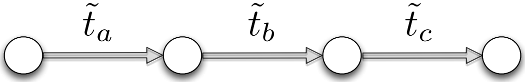

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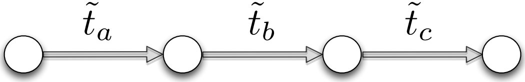

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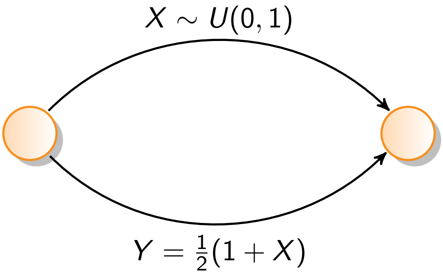
DRAWBACKS:

- Lack of monotonicity
- Lack of additive consistency
- Bellman's principle fails: finding optimal paths is hard

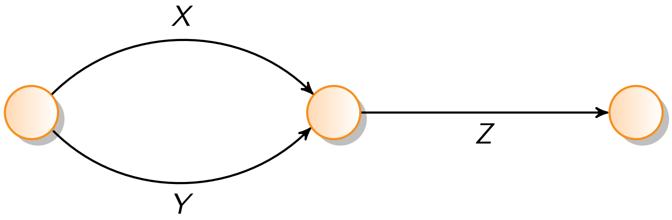
Lack of monotonicity



Hence $Y > X$ a.s. but for $\gamma = 12$ we have

$$\Phi_\gamma(Y) = \frac{5}{4} < \Phi_\gamma(X) = \frac{3}{2}$$

Lack of additive consistency



If $\Phi(X) \leq \Phi(Y)$ and Z independent... then $\Phi(X+Z) \leq \Phi(Y+Z)$?

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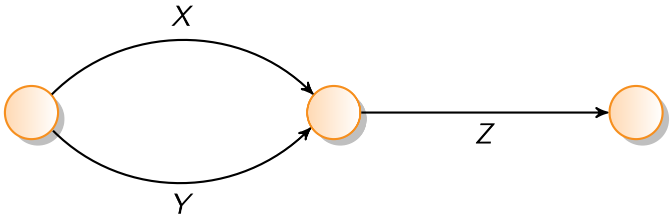


If $\Phi(X) \leq \Phi(Y)$ and Z independent... then $\Phi(X+Z) \leq \Phi(Y+Z)$?

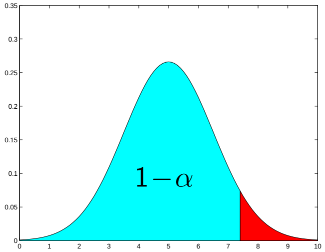
Not necessarily! Consider $\gamma = 1$ and

$$X \sim N(10.9, 1) \quad ; \quad Y \sim N(10, 4) \quad ; \quad Z \sim N(10, 1)$$

$$\Phi(X) = 11.9 \quad < \quad \Phi(Y) = 12.0$$
$$\Phi(X+Z) = 22.3 \quad > \quad \Phi(Y+Z) = 22.2$$

How to measure risk: Value-at-Risk (...late 1980's)

$$\Phi(X) = VaR_\alpha(X) = F_X^{-1}(1-\alpha) = (1-\alpha)\text{-percentile}$$



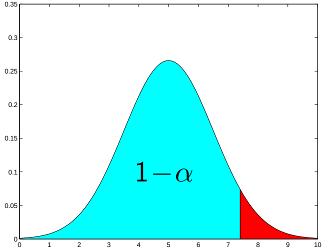
It is monotone. Coincides with mean-stdev for Normal distributions ⇒

- Not additive consistent
- Bellman's principle fails: finding optimal paths is hard

How to measure risk: Average Value-at-Risk

(Artzner *et al.* 1999; Rockafellar and Uryasev 2000)

$$\Phi(X) = AVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_t(X)dt = \mathbb{E}[X|X \geq VaR_\alpha(X)]$$



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How to measure risk: Coherent risk measures
(Artzner *et al.* 1999)

A map $\Phi : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ is a **risk measure** if $\Phi(0) = 0$ and it is

- *Monotone:* $X \leq Y$ a.s. $\Rightarrow \Phi(X) \leq \Phi(Y)$
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coherent: if Φ is sublinear

convex: if Φ is convex

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Remark:

- Trans. inv. $\Leftrightarrow$ $\Phi(m)=m$ and $\Phi(X)\leq\Phi(Y) \Rightarrow \Phi(X+m)\leq\Phi(Y+m)$
- Under translation invariance "additive $\Leftrightarrow$ additive consistent"

How to measure risk: Expected utility

(Bernoulli 1738; Kolmogorov 1930; Nagumo 1931; de Finetti 1931; von Neuman-Morgenstern 1947)

For $c : \mathbb{R} \to \mathbb{R}$ increasing the *expected utility map*

$$\Phi_c(X) = c^{-1}(\mathbb{E}\, c(X))$$

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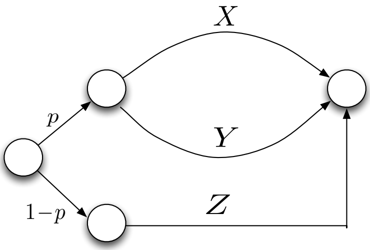

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REMARKS:

- These properties characterize expected utility preferences
- Risk-aversion $\equiv$ exaggerate effect of bad events — $c(\cdot)$ convex
- But Φ_c is not translation invariant, hence not a risk measure !

Entropic Risk Measures

Theorem

The only expected utility maps Φ_c that are translation invariant — and hence risk measures — are the β-entropic risk measures

$$\Phi_\beta(X) = \frac{1}{\beta}\ln(\mathbb{E}\, e^{\beta X}).$$

associated with $c(x) = e^{\beta x}$ where $-\infty < \beta < \infty$.

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- Φ_β is also additive and hence additive consistent
- For $\beta \geq 0$ it is convex and risk averse
- Coherent only for $\Phi_0(X) = \mathbb{E}(X)$

Sketch of Proof

From $\Phi_c(m+zB_p) = m+\Phi_c(zB_p)$ with B_p Bernoulli we get differentiability of $c(\cdot)$ and the functional equation

$$c'(0)[c(m+z)-c(m)] = c'(m)[c(z) - c(0)]$$

whose solutions are $c(x) = e^{\beta x}$ (up to an affine transformation). $\square$

How to measure risk: Dual theory of choice
(Allais 1953; Yaari 1987)

Let $h : [0,1] \to [0,1]$ increasing, $h(0) = 0, h(1) = 1$. The *h-distorted risk measure* is defined by

$$\Phi^h(X) = \mathbb{E}(X^h)$$

where X^h is a random variable with distribution

$$\mathbb{P}(X^h \leq x) = h(\mathbb{P}(X \leq x)).$$

Risk-aversion $\equiv$ exaggerate the probability of bad events — $h(s) \leq s$

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These measures are characterized by the *dual independence axiom*:

$$(DIA) \qquad \Phi(X) \leq \Phi(Y) \Rightarrow \Phi(\alpha X + (1-\alpha)Z) \leq \Phi(\alpha Y + (1-\alpha)Z)$$

for all X, Y, Z pairwise co-monotonic.

How to measure risk: Combine utility & distortion

(Allais 1953; Schmeidler 1989; Quiggin 1993; Wakker 1994)

Given a utility function $c : \mathbb{R} \to \mathbb{R}$ and a distortion map $h : [0,1] \to [0,1]$

$$\Phi_c^h(X) = c^{-1}(\mathbb{E}\, c(X^h)).$$

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Characterized by weaker independence axiom: tradeoff consistency.

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Theorem

The only maps Φ_c^h that are additive consistent are the entropic risk measures Φ_β

REMARK: Under smoothness assumptions this result was obtained by Luan'2001, Heilpern'2003, Goovaerts-Kaas-Laeven-Tang'2010

Sketch of Proof

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Step 1: From $\Phi_c^h(m + zB_p) = m + \Phi_c^h(zB_p)$ we get

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Step 2: From $\Phi_c^h(zB_p + zB_q) = \Phi_c^h(zB_p) + \Phi_c^h(zB_q)$ we get

$$\begin{aligned} h(pq) &= h(p)h(q) \\ h(p) + h(q) &= h(p)h(q) + h(1 - \bar{p}\bar{q}) \end{aligned}$$

with unique solution $h(s) = s$. □

Computing entropic optimal paths

Let $G = (V, A)$ with all the $\tilde{t}_a$'s independent. By additive consistency, the risk of the random time $X = \sum_{a \in r} \tilde{t}_a$ of a path r satisfies

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COMMENT: Dependent case yields a stochastic dynamic programming recursion solved by conditional expectation

$$\Phi_\beta(X + Y) = \Phi_\beta(X + \Phi_\beta(Y|X)).$$

Routing games with entropic risk averse players

If the distribution $\tilde{t}_a \sim F(v_a)$ depends on the load v_a of link a so that $\Phi_\beta(\tilde{t}_a) = g_a(v_a)$ is an increasing function of v_a, then

- non-atomic equilibrium falls into Wardrop's framework
- the atomic case is a special case of Rosenthal's framework

Dynamic risk measures & consistency

Consider a sequence of payoffs $X_t \in \mathcal{Z}_t = L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ adapted to a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T \subseteq \mathcal{F}$.

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A sequence of **conditional risk measures** $\rho_t : \mathcal{Z}_t \to \mathcal{Z}_{t-1}$ which are

- *monotone*: $X \leq Y \Rightarrow \rho_t(X) \leq \rho_t(Y)$
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is called **dynamically consistent** if the nested risk transition maps

$$R_t^T(X_t, \dots, X_T) = \rho_t(X_t + \rho_{t+1}(X_{t+1} + \cdots + \rho_T(X_T)))$$

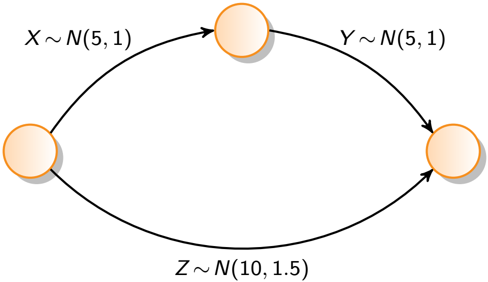
are such that

$$R_t^T(X_t, \dots, X_T) \leq R_t^T(Y_t, \dots, Y_T)$$

$$\Downarrow$$

$$R_{t-1}^T(Z, X_t, \dots, X_T) \leq R_{t-1}^T(Z, Y_t, \dots, Y_T)$$

Routing stages & recursive AVaR ?



$$\rho_1(X+\rho_2(Y)) > \rho_1(Z)$$

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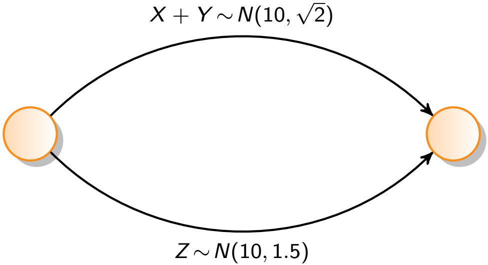

$$\rho_1(X+Y) < \rho_1(Z)$$

This is the end... !