Bundle methods for stochastic programs
Proximal bundle method

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General formulation

In this part of the course we will focus on efficient optimization methods to solve convex programs of the form

$$\min f(x) \quad \text{s.t.} \quad x \in X,$$

with

- $f : \mathbb{R}^n \to \mathbb{R}$ a convex but nonsmooth function

- $X \subset \mathbb{R}^n$ a convex set (e.g. $X = \{ x \in \mathbb{R}_+^n : Ax = b \}, X = \mathbb{R}^n$)

This formulation covers many practical optimization problems, for instance

- Two-stage stochastic programming problems

- Multistage stochastic programming problems
Two-stage stochastic linear programming

In two-stage stochastic linear programming problems with finitely many scenarios $\xi^i = (q^i, T^i, W^i, h^i)$ we wish to solve the high dimensional LP

$$\begin{cases}
\min & c^\top x + \sum_{i=1}^N p_i [q^i \top y^i] \\
\text{s.t.} & Ax = b, \ x \geq 0 \\
& T^i x + W^i y^i = h^i, \ y^i \geq 0, \ i = 1, \ldots, N
\end{cases}$$
TWO-STAGE STOCHASTIC LINEAR PROGRAMMING

In two-stage stochastic linear programming problems with finitely many scenarios $\xi^i = (q^i, T^i, W^i, h^i)$ we wish to solve the high dimensional LP

$$\begin{align*}
\min & \quad c^\top x + \sum_{i=1}^{N} p_i [q^i]^\top y^i \\
\text{s.t.} & \quad Ax = b, \ x \geq 0 \\
& \quad T^i x + W^i y^i = h^i, \ y^i \geq 0, \ i = 1, \ldots, N
\end{align*}$$

TWO-STAGE DECOMPOSITION

$$\begin{align*}
\min & \quad f(x) \quad \text{s.t.} \quad x \in X, \quad \text{with} \quad f(x) := c^\top x + \sum_{i=1}^{N} p_i Q(x, \xi^i), \\
Q(x, \xi) = & \begin{cases}
\min & \quad q^\top y \\
\text{s.t.} & \quad Wy = h - Tx \\
& \quad y \geq 0.
\end{cases}
\end{align*}$$

We know that $g = c - \sum_{i=1}^{N} p_i T^i \pi^i \in \partial f(x)$, where $\pi^i$ is a dual solution of $Q(x, \xi^i)$.
Some elements of the data $\xi = (c_t, B_t, A_t, b_t)$ depend on uncertainties. By assuming finitely many scenarios and dualizing the nonanticipativity constraints (that can be written as $Gx = 0$) we get
Multistage stochastic linear programs

(See Lecture 17)

Dual problem

\[ \min_u f(u), \quad \text{with} \quad f(u) := - \sum_{i=1}^{N} D^i(u) \]

\[ D^i(u) := \begin{cases} \min_{x^i} & p_i \sum_{t=1}^{T} (c_t^i) \top x_t^i + u \top G^i x^i, \\ \text{s.t.} & A_1 x_1 = b_1, \\ & B_t^i x_{t-1}^i + A_t^i x_t^i = b_t^k, \quad t = 2, \ldots, T, \\ & x_t^i \geq 0. \end{cases} \]

Computing \( f(u) \) for each given \( u \) amounts to solving \( N \) LPs.

We know that \( g = -Gx(u) \in \partial f(u) \), where \( x(u) = (x^1(u), \ldots, x^N(u)) \) and \( x^i(u) \) is a solution of \( D^i(u) \)
Let’s stick with the more compact and general formulation

$$\min f(x) \quad \text{s.t.} \quad x \in X,$$

with $f : \mathbb{R}^n \to \mathbb{R}$ a convex but nonsmooth function and $X \subset \mathbb{R}^n$ a convex set.

We’ll assume the availability of an oracle providing us with first-order information on $f$:

$$x \xrightarrow{\text{Oracle}} \begin{cases} \text{function value} & f(x) \\ \text{subgradient} & g \in \partial f(x) \end{cases}$$

In stochastic programming, the oracle should be smart enough to use parallel computing:

- the oracle consists of solving $N$ optimization subproblems to compute $f(x)$ and a subgradient $g$
- most of time dedicate to minimize $f$ is spent in the oracle!

Therefore, subgradient and (pure) cutting-plane methods are not very efficient\(^1\)...

\(^1\)These methods require, in general, many oracle calls.
**Cutting-plane method**

Consider the problem

\[
\min_{x \in X} f(x)
\]

and suppose that \( X \) is a compact set.

**Algorithm**

1. Given \( x_0 \in X \), call the oracle to compute \( f(x_0) \) and \( g_0 \in \partial f(x_0) \). Set \( f_{0}^{\text{up}} = f(x_0) \) and \( k = 0 \)
2. (iterate) Find \( x_{k+1} = \arg \min_{x \in X} \tilde{f}_k(x) \). Let \( f_{k}^{\text{low}} = \tilde{f}_k(x_{k+1}) \).
3. (stopping test) If \( f_{k}^{\text{up}} - f_{k}^{\text{low}} \) is small enough, stop.
4. (oracle) Compute \( f(x_{k+1}) \), \( g_{k+1} \in \partial f(x_k) \) and set \( f_{k+1}^{\text{up}} = \min\{f(x_{k+1}), f_{k}^{\text{up}}\} \).
5. (loop) Set \( k \leftarrow k + 1 \) and go back to Step 2.

**Cutting-plane model**

\[
\tilde{f}_k(\cdot) = \max_{j=1,\ldots,k} \{ f(x_j) + g_j^\top (\cdot - x_j) \} 
\]
Cutting-plane method

\[ f(x) \]

\[ X \]

\[ x^1 \]
Cutting-plane method
CUTTING-PLANE METHOD

$f(x)$
Cutting-plane method
Cutting-plane method

\[ f(x) \]

\[ x^1 \quad x^5 \quad x^4 \quad x^3 \quad x^2 \]
CUTTING-PLANE METHOD

\[ f(x) \]

6 iterações!
The method requires solving a LP at each iteration

\[ x_{k+1} = \arg \min_{x \in X} \tilde{f}_k(x), \quad \tilde{f}_k(\cdot) = \max_{j=1,\ldots,k} \{ f(x_j) + g_j^\top (\cdot - x_j) \} \]

that is equivalent to

\[
\begin{aligned}
\min_{x,r} & \quad r \\
\text{s.t.} & \quad f(x_j) + g_j^\top (x - x_j) \leq r, \quad j = 1, \ldots, k \\
& \quad x \in X, \ r \in \mathbb{R}.
\end{aligned}
\]

A new constraint is added at each iteration!
Cutting-plane method

Pros × Cons

✔️ only computes a single subgradient per iteration
✔️ easy to code
✔️ easy and reliable stopping test

❌ $f(x_{k+1}) \not\leq f(x_k)$ (it is not a descent method)
❌ instable and has low convergence rate
❌ requires compactness of the feasible set
❌ doesn’t exploit good starting points
❌ subproblem becomes heavier and heavier...

The Regularized Decomposition Method (1986) for 2-SLP address some of the above drawbacks.

Regularized Decomposition Method is just a particular case of (proximal) Bundle Methods!
Main ingredients

(i) a convex model $f^M_k \leq f$ (eg. cutting-plane model)
(ii) a stability center $\hat{x}_k$ (eg.: the best point so far)
(iii) a parameter $t_k$ (or $f_{lev}^k$) to be updated at every iteration

The next trial point $x_{k+1}$ of a bundle method depends on the above 3 ingredients, whose organization define different methods:

**Proximal bundle method** ($t_k > 0$)

$$x_{k+1} := \arg\min \left\{ f^M_k(x) + \frac{1}{2t_k} \|x - \hat{x}_k\|^2 : x \in X \right\} .$$

**Level bundle method** ($f_{lev}^k \in \mathcal{R}$)

$$x_{k+1} := \arg\min \left\{ \frac{1}{2} \|x - \hat{x}_k\|^2 : f^M_k(x) \leq f_{lev}^k, x \in X \right\} .$$

Today we focus on proximal bundle method!
Proximal bundle method

\[ f^M \equiv \tilde{f}, \quad x_{k+1} := \arg \min \left\{ \tilde{f}_k(x) + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 : x \in X \right\} \]
Proximal bundle method

\[ f^M \equiv \tilde{f}, \quad x_{k+1} := \arg \min \left\{ \tilde{f}_k(x) + \frac{1}{2t_k} \|x - \hat{x}_k\|^2 : x \in X \right\} \]
Proximal bundle method

\[ f^M \equiv \tilde{f}, \quad x_{k+1} := \arg \min \left\{ \tilde{f}_k(x) + \frac{1}{2t_k} \|x - \hat{x}_k\|^2 : x \in X \right\} \]
Proximal bundle method

\[ f^M = \tilde{f}, \quad x_{k+1} := \arg \min \left\{ \tilde{f}_k(x) + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 : x \in X \right\} \]
Proximal bundle method

\[ f^M \equiv \tilde{f}, \quad x_{k+1} := \arg \min \left\{ \tilde{f}_k(x) + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 : x \in X \right\} \]
Proximal bundle method

Pros × Cons

- only computes a single subgradient per iteration
- easy and reliable stopping test
- stable
- does not require $X$ to be compact
- it is a descent method
- exploit good-quality initial points
- subproblem defining $x_{k+1}$ can be kept small
Proximal bundle method

Pros × Cons

- only computes a single subgradient per iteration
- easy and reliable stopping test
- stable
- does not require $X$ to be compact
- it is a descent method
- exploit good-quality initial points
- subproblem defining $x_{k+1}$ can be kept small

- convergence analysis is more involving...
Proximal bundle method

Let’s consider a more economical model:

\[ f^M_k(x) := \max_{j \in B_k} \{ f(x_j) + g_j^\top(x - x_j) \} \]

- The cutting-plane method takes \( B_k := \{1, 2, \ldots, k\} \). We will consider \( B_k \subset \{1, 2, \ldots, k\} \) (or something a bit different)

- The method generates a sequence of trial points \( \{x_k\} \subset X \) by solving a QP:

\[ x_{k+1} := \arg \min \left\{ f^M_k(x) + \frac{1}{2t_k} \|x - \hat{x}_k\|^2 : x \in X \right\} . \]
The QP

$$\min \left\{ f_k^M(x) + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 : x \in X \right\}$$

can be rewritten as

$$\left\{ \begin{array}{l}
\min_{x, r} \quad r + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 \\
\text{s.a} \quad f(x_j) + g_j^\top (x - x_j) \leq r, \quad j \in B_k \\
x \in X, \quad r \in \mathbb{R}
\end{array} \right.$$  

We can apply specialized softwares.
**Proximal bundle method**

\[ x_{k+1} := \arg \min \left\{ f_k^M(x) + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 : x \in X \right\} . \]

A rule decides when to update the stability center \( \hat{x}_k \). Such rule depends on the predicted decrease by the model \( f_k^M \)

\[ v_k = f(\hat{x}_k) - f_k^M(x_{k+1}) \]

and a constant \( \kappa \in (0, 1) \):

- **Serious step:** if \( f(x_{k+1}) \leq f(\hat{x}_k) - \kappa v_k \), then
  \[ \hat{x}_{k+1} \leftarrow x_{k+1} \]

- **Null step:** if \( f(x_{k+1}) > f(\hat{x}_k) - \kappa v_k \), then
  \[ \hat{x}_{k+1} \leftarrow \hat{x}_k \]

The serious-step sequence \( \{ \hat{x}_k \} \) is a subsequence of \( \{ x_k \} \).
**Lemma**

Suppose that $X$ is a polyhedron or $ri(X) \neq \emptyset$. Then

$$x_{k+1} = \hat{x}_k - t_k \hat{g}_k \quad \text{com} \quad \hat{g}_k = p^k_f + p^k_X,$$

where $p^k_f \in \partial f^M_k(x_{k+1})$ and $p^k_X \in \partial i_X(x_{k+1})$.

($i_X$ is the indicator function of $X$.)

Furthermore, the affine function

$$f^L_{ka}(x) := f^M_k(x_{k+1}) + \langle \hat{g}_k, x - x_{k+1} \rangle$$

is a lower approximation for the model $f^M_k$:

$$f^L_{ka}(x) \leq f^M_k(x) \quad \forall x \in X.$$
OPTIMALITY MEASURE

PROPOSITION
Let the predicted decrease and aggregate linearization error defined by

\[ v_k := f(\hat{x}_k) - f_k^M(x_{k+1}) \quad \text{and} \quad \hat{e}_k := f(\hat{x}_k) - f_k^L(\hat{x}_k). \]

Then,

\[ \hat{e}_k \geq 0, \quad \hat{e}_k + t_k \|\hat{g}_k\|^2 = v_k \geq 0 \quad \text{for all} \quad k. \]

Furthermore

\[ f(\hat{x}_k) \leq f(x) + \hat{e}_k + \|\hat{g}_k\|\|\hat{x}_k - x\| \quad \text{for all} \quad x \in X \quad \text{and} \quad k. \]

If \((\hat{e}_k, \hat{g}_k) = 0\), then \(\hat{x}_k\) is solution to the problem
Algorithm: proximal bundle method

$$f^M_k(x) = \max_{j \in B_k} \{ f(x_j) + g_j^\top (x - x_j) \}, \quad x_{k+1} = \arg \min \left\{ f^M_k(x) + \frac{1}{2t_k} \| x - \hat{x}_k \|^2 : x \in X \right\}$$

**Step 0.** Choose $\kappa \in (0, 1)$, $t_1 \geq t_{\text{min}} > 0$, $x_1 \in X$ and tolerance $\text{tol} > 0$. Call the oracle to compute $(f(x_1), g_1)$. Define $\hat{x}_1 \leftarrow x_1$, $k \leftarrow 1$, $B_1 \leftarrow \{1\}$,

**Step 1.** Solve the QP to obtain $x_{k+1}$. Define $\hat{g}_k \leftarrow (\hat{x}_k - x_{k+1})/t_k$, $v_k \leftarrow f(\hat{x}_k) - \tilde{f}_k(x_{k+1})$, and $\hat{e}_k \leftarrow v_k - t_k \| \hat{g}_k \|^2$

**Step 2.** If $\hat{e}_k \leq \text{tol}$ and $\| \hat{g}_k \| \leq \text{tol}$, stop: $\hat{x}_k$ is an approximate solution

**Step 3.** Call the oracle to obtain $(f(x_{k+1}), g_{k+1})$

Serious step. **If** $f(x_{k+1}) \leq f(\hat{x}_k) - \kappa v_k$, **then** $\hat{x}_{k+1} \leftarrow x_{k+1}$

and choose $t_{k+1} \geq t_k$.

Null step. **Otherwise,** define $\hat{x}_{k+1} \leftarrow \hat{x}_k$ and choose $t_{k+1} \in [t_{\text{min}}, t_k]$

**Step 4.** Choose $B_{k+1} \supset \{k + 1, k^a\}$

Set $k \leftarrow k + 1$ and go back to Step 1.
Some Comments

- Only 2 linearizations are required: $f_k^L$ and $f_{k^a}^L$, i.e.,

$$\mathcal{B}_{k+1} = \{k + 1, k^a\} \text{ suffices!}$$

- the prox-parameter $t_k$ is non-increasing along null steps

- a simple heuristic to update the prox-parameter is the following

  - compute $t_{aux} := t_k \left(1 + \frac{(g_{k+1} - g_k)^\top (x_{k+1} - x_k)}{\|g_{k+1} - g_k\|^2}\right)$

  - if null step: $t_{k+1} \leftarrow \min\{t_k, \max\{t_{aux}, t_k/2, t_{min}\}\}$

  - if serious step: $t_{k+1} \leftarrow \max\{t_k, \min\{t_{aux}, 10t_k\}\}$

- it is advisable to consider different tolerances for the measures $\hat{e}_k$ and $\hat{g}_k$

- the sequence $\{f(\hat{x}_k)\}$ is non-increasing

- any accumulation point of $\{\hat{x}_k\}$ is a solution to the problem