Multistage stochastic linear programming problems

Block separable recourse

Welington de Oliveira

BAS Lecture 16, May 3, 2016, IMPA

News

Nested Decomposition - convergence analysis

Block separable recourse

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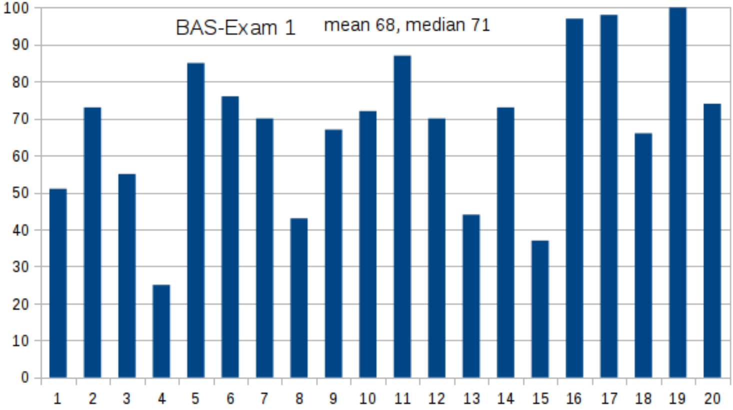
BAS-Exam 1
mean 68, median 71
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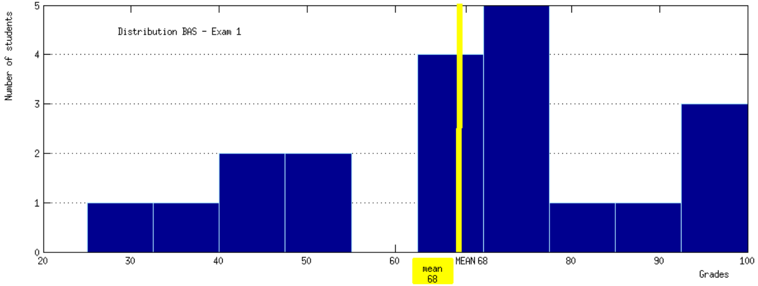
Number of students
Distribution BAS - Exam 1
0
1
2
3
4
5
20
30
40
50
60
MEAN 68
80
90
100
mean
68
Grades

Exercises

Second list of exercises is available!

Deadline: 02/06/2016

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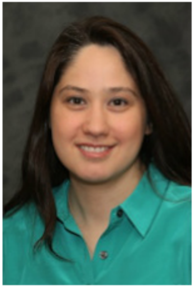

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Today ◀ ▶ May 2016 ▾ Print Week **Month** Agenda

Sun	Mon	Tue	Wed	Thu	Fri	Sat
May 1	2	3 1:30pm SVAN -- Bas	4	5 1:30pm SVAN -- Bas	6	
8	9 1:30pm MC1: Scena 3:30pm MC1: Scena	10 1:30pm SVAN -- Bas	11 10:30am MC1: Scen 1:30pm MC1: Scena 3:30pm MC1: Scena	12 1:30pm SVAN -- Bas	13 10:30am MC1: Scen 1:30pm MC1: Scena 3:30pm MC1: Scena	1
15	16 10:30am MC3: Stoch 1:30pm MC3: Stocha 3:30pm MC2: Equilib	17 9am MC3: Stochasti 1:30pm SVAN -- Bas 3:30pm MC2: Equilib	18 9am MC3: Stochasti 10:30am MC3: Stoch 1:30pm MC2: Equilib	19 9am MC3: Stochasti 1:30pm SVAN -- Bas 3:30pm MC2: Equilib	20 9am MC3: Stochasti 10:30am MC3: Stoch	2
22	23	24 1:30pm SVAN -- Bas	25	26 1:30pm SVAN -- Bas	27	2

Multistage stochastic linear programs - T-SLP

Nested formulation

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}\left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \mathbb{E}\left[\cdots + \mathbb{E}[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T]\right]\right]$$

- Some elements of the data $\xi = (c_t, B_t, A_t, b_t)$ depend on uncertainties.

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi_{[T]}) := \min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T$$

- At stages $t = 2, \ldots, T-1$

$$Q_t(x_{t-1}, \xi_{[t]}) := \min_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$$

- Stage $t = 1$

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathcal{Q}_2(x_1, \xi_{[1]})$$

Recourse function

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) := \mathbb{E}_{|\xi_{[t]}} \left[Q_{t+1}(x_t, \xi_{[t+1]}) \right]$$

Dynamic programming formulation

Scenario tree

- Stage $t = T$

$$Q_T(x_{T-1}, \xi^\iota_{[T]}) := \min_{\substack{B^\iota_T x^{a(\iota)}_{T-1} + A^\iota_T x_T = b^\iota_T \\ x_T \geq 0}} {c^\iota_T}^\top x_T$$

- At stages $t = 2, \ldots, T-1$

$$\underline{Q_t}(x_{t-1}, \xi^\iota_{[t]}) := \min_{\substack{B^\iota_t x^{a(\iota)}_{t-1} + A^\iota_t x_t = b^\iota_t \\ x_t \geq 0}} {c^\iota_t}^\top x_t + \check{\mathcal{Q}}_{t+1}(x_t, \xi^\iota_{[t]})$$

- Stage $t = 1$

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \check{\mathcal{Q}}_2(x_2, \xi_{[1]})$$

Cutting-plane model

$$\check{\mathcal{Q}}_{t+1}(x_t, \xi^\iota_{[t]}) := \sum_{j \in C_\iota} p^{(j)} \left[\underline{Q_{t+1}}(x_t, \xi^j_{[t+1]}) \right]$$

Cutting-plane approximation

- Stage $t = T$

$$Q_T(x_{T-1}^k, \xi_{[T]}) := \min_{\substack{B_T x_{T-1}^k + A_T x_T = b_T \\ x_T \geq 0}} c_T{}^\top x_T$$

- At stages $t = 2, \ldots, T-1$

$$\underline{Q}_t(x_{t-1}^k, \xi_{[t]}) := \begin{cases} \min\limits_{x_t \geq 0, r_{t+1}} & c_t{}^\top x_t + r_{t+1} \\ \text{s.t.} & B_t x_{t-1}^k + A_t x_t = b_t \\ & r_{t+1} \geq \alpha_{t+1}^j + \beta_{t+1}^j x_t \quad j = 1, \ldots, k \end{cases}$$

- Stage $t = 1$

$$\underline{z}^k := \begin{cases} \min\limits_{x_1 \geq 0, r_2} & c_1^\top x_1 + r_2 \\ \text{s.t.} & A_1 x_1 = b_1 \\ & r_2 \geq \alpha_2^j + \beta_2^j x_1 \quad j = 1, \ldots, k \end{cases}$$

Computing cuts

- At stages $t = 2, \dots, T-1$

$$\underline{Q_t}(x_{t-1}^k, \xi_{[t]}) := \begin{cases} \min\limits_{x_t \geq 0, r_{t+1}} & c_t{}^\top x_t + r_{t+1} \\ \text{s.t.} & B_t x_{t-1}^k + A_t x_t = b_t & & (\pi_t) \\ & r_{t+1} \geq \alpha_{t+1}^j + \beta_{t+1}^j x_t & j = 1, \dots, k & (\rho_t^j) \end{cases}$$

- Cuts $(t = T)$

$$\alpha_T^k := \mathbb{E}_{|\xi_{[T-1]}}[b_T^\top \pi_T^k] \quad \text{and} \quad \beta_T^k := -\mathbb{E}_{|\xi_{[T-1]}}[B_T^\top \pi_T^k]$$

- Cuts $(t = T-1, \dots, 2)$

$$\alpha_t^k := \mathbb{E}_{|\xi_{[t-1]}}[b_t^\top \pi_t^k + \sum_{j=1}^k \alpha_{t+1}^j \rho_t^j] \quad \text{and} \quad \beta_t^k := -\mathbb{E}_{|\xi_{[t-1]}}[B_t^\top \pi_t^k]$$

$$\begin{aligned} \check{\mathcal{Q}}_{t+1}(x_t, \xi_{[t]}^\iota) &= \textstyle\sum_{j \in C_\iota} p^{(j)} \left[\underline{Q_{t+1}}(x_t, \xi_{[t+1]}^j)\right] \\ &= \max_{j=1,\dots,k}\{\alpha_{t+1}^k + \beta_{t+1}^k{}^\top x_t\} \end{aligned}$$

Algorithm - Nested decomposition

stages $t = 2, \dots, T-1$

$$\underline{Q}_t(x_{t-1}^k, \xi_{[t]}) := \begin{cases} \min_{x_t \geq 0, r_{t+1}} & c_t^\top x_t + r_{t+1} \\ \text{s.t.} & B_t x_{t-1}^k + A_t x_t = b_t & & (\pi_t) \\ & r_{t+1} \geq \alpha_{t+1}^j + \beta_{t+1}^j x_t & j = 1, \dots, k & (\rho_t^j) \end{cases}$$

- **Step 0: initialization**. Define $k = 1$ and add the constraint $r_t = 0$ in all LPs $\underline{Q}_t$, $t = 2, \dots, T-1$. Compute $\underline{z}^1$ and let its solution be x_1^1.
- **Step 1: forward**. For t=2,...,T, solve the LP $\underline{Q}_t$ to obtain $x_t^k := x_t^k(\xi_{[t]})$. Define $\bar{z}^k := \mathbb{E}[\sum_{t=1}^T c_t^\top x_t^k]$.
- **Step 2: backward**. Compute α_T^k and β_T^k. Set $t = T$. Loop:
  - While $t > 2$
  - $t \leftarrow t - 1$
  - solve the LP $\underline{Q}_t(x_{t-1}^k, \xi_{[t]})$
  - Compute α_t^k and β_t^k

  Compute $\underline{z}^k$ and let its solution be x_1^{k+1}.
- **Step 3: Stopping test**. If $\bar{z}^k - \underline{z}^k \leq \epsilon$, stop. Otherwise set $k \leftarrow k + 1$ and go back to Step 1.

Convergence analysis

Assumptions

- The set of nodes Ω_t has finitely many elements, $t = 1, \ldots, T$
- the problem has recourse relatively complete (for simplicity, only)
- the feasible set, in each stage $t = 1, \ldots, T$, is compact

Lemma

$\check{\mathcal{Q}}_t^k(x_{t-1}, \xi_{[t-1]}) \leq \mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}) \quad \forall\, x_{t-1}$ *and* $\forall t = 2, \ldots, T$

Theorem

The Nested Decomposition converges finitely to an optimal solution of the considered T-SLP.

Block separable recourse

If the T-SLP problem has block separable recourse, then a more efficient algorithm might be employed (this will, of course, depend on the application).

Definition

A T-SLP has block separable recourse if for all stage $t = 1, \dots, T$ and all ξ, the decision vectors, x_t, can be written as $x_t = (w_t, y_t)$ where w_t represents aggregate level decisions and y_t represents detailed level (local) decisions. The constraints also follow these partitions:

- The stage t cost is $c_t^\top x_t = c_t^{w\top} w_t + c_t^{y\top} y_t$
- The matrices in the coupling constraint $B_t x_{t-1} + A_t x_t = b_t$ are given by

$$B_t = \begin{pmatrix} T_t & 0 \\ S_t & 0 \end{pmatrix} \quad A_t = \begin{pmatrix} W_t & 0 \\ 0 & D_t \end{pmatrix} \quad b_t = \begin{pmatrix} h_t \\ d_t \end{pmatrix}$$

Block separable recourse

$$x_t = (w_t, y_t) \qquad c_t^\top x_t = c_t^{w\,\top} w_t + c_t^{y\,\top} y_t$$

In this manner

$$B_t x_{t-1} + A_t x_t = b_t \quad \Longleftrightarrow \quad \begin{cases} T_t w_{t-1} + W_t w_t = h_t \\ S_t w_{t-1} + D_t y_t = d_t \end{cases}$$

and the cost-to-go function

$$Q_t(x_{t-1}, \xi_{[t]}) := \min_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$$

becomes the sum of two quantities

$$Q_t(x_{t-1}, \xi_{[t]}) = Q_t^w(w_{t-1}, \xi_{[t]}) + Q_t^y(w_{t-1}, \xi_{[t]})$$

with

$$Q_t^w(w_{t-1}, \xi_{[t]}) := \min_{\substack{T_t w_{t-1} + W_t w_t = h_t \\ w_t \geq 0}} c_t^{w\,\top} w_t + \mathcal{Q}_{t+1}(w_t, \xi_{[t]})$$

and

$$Q_t^y(w_{t-1}, \xi_{[t]}) := \min_{\substack{S_t w_{t-1} + D_t y_t = d_t \\ y_t \geq 0}} c_t^{y\,\top} y_t$$

Block separable recourse

The great advantage of block separability is that we need not consider nesting among the detailed level decisions. In this way, the w variables can all be pulled together into a first stage of aggregate level decisions.

$$\begin{array}{ll} \min_{x_1,w} & c_1^\top x_1 + \mathbb{E}[{c_2^w}^\top w_2 + \cdots + {c_T^w}^\top w_T] + \mathbb{E}[\sum_{t=2}^T Q_t^y(w_{t-1}, \xi_{[t]})] \\ \text{s.t.} & A_1 x_1 = b_1 \\ & T_t w_{t-1} + W_t w_t = h_t, \quad t = 2, \ldots, T \quad a.s. \\ & x_1, w \geq 0 \end{array}$$

with

$$Q_t^y(w_{t-1}, \xi_{[t]}) := \min_{\substack{S_t w_{t-1} + D_t y_t = d_t \\ y_t \geq 0}} {c_t^y}^\top y_t$$

Block separable recourse

With finitely many scenarios

$$\min_{z \in Z} \bar{c}^\top z + \mathcal{Q}(z)$$

with Z a polyhedral set, z containing all the node decisions w_t^ι and x_1 and

$$\mathcal{Q}(z) = \sum_{t=2}^{T} \sum_{\iota \in \Omega_t} p^{(\iota)} Q_t^y(z, \xi^\iota)$$

$$Q_t^y(z, \xi^\iota) = \min_{\substack{S_t^\iota w_{t-1}^{a(\iota)} + D_t^\iota y_t = d_t^\iota \\ y_t \geq 0}} c_t^{y,\iota\top} y_t$$

This is a convex programming problem and a subgradient of $\mathcal{Q}$ is computable!

Block separable recourse

Cutting-plane method

The problem

$$\min_{z \in Z} f(z), \quad \text{with} \quad f(z) = \bar{c}^\top z + \mathcal{Q}(z)$$

Oracle

$$z^\ell \longrightarrow \boxed{\textbf{oracle}} \longrightarrow \left[\begin{array}{l} f(z^\ell) = \bar{c}^\top z^\ell + \mathcal{Q}(z^\ell) \\ g^\ell \in \partial f(z^\ell) \end{array} \right.$$

Cutting-plane model

$$\check{f}_\ell(z) := \max_{j=1,\ldots,\ell} \{ f(z^j) + \langle g^j, x - x^j \rangle \}$$

Next iterate

$$z^{\ell+1} \in \arg\min_{z \in Z} \check{f}_\ell(z)$$

Cutting-plane algorithm

- **Step 0: inicialization**. Choose tol > 0, $z^0 \in Z$ and call the oracle to compute $f(z^0)$ and $g^0 \in f(z^0)$. Set $f_0^{\text{up}} = f(z^0)$ and $\ell = 0$

- **Step 1: next iterate**. Compute

$$z^{\ell+1} \in \arg\min_{z \in Z} \check{f}_\ell(z)$$

and let $f_\ell^{\text{low}} = \check{f}_\ell(z^{\ell+1})$.

- **Step 2: stopping test**. Define $\Delta_\ell = f_\ell^{\text{up}} - f_\ell^{\text{low}}$. If $\Delta_\ell \leq$ tol, stop

- **Step 3: oracle call**. Compute $f(z^{\ell+1})$ and $g^{\ell+1} \in f(z^{\ell+1})$ and set $f_{\ell+1}^{\text{up}} = \min\{f_\ell^{\text{up}}, f(z^{\ell+1})\}$.

- **Step 4: loop**. Set $\ell = \ell + 1$ and go back to Step 1.

Convergence analysis

Theorem

Let tol > 0 *be given and suppose that* Z *is compact. Then the cutting-plane algorithm determines* $\Delta_\ell \leq \text{tol}$ *in finitely many iterations. Furthermore, the point* $\bar{z}$ *yielding* $f_\ell^{\text{up}} = f(\bar{z})$ *is a* tol*-solution to the block separable T-SLP.*

Convergence analysis

Theorem

Let tol > 0 *be given and suppose that* Z *is compact. Then the cutting-plane algorithm determines* $\Delta_\ell \leq$ tol *in finitely many iterations. Furthermore, the point* $\bar{z}$ *yielding* $f_\ell^{\text{up}} = f(\bar{z})$ *is a* tol*-solution to the block separable T-SLP.*

In fact, the result also holds if:

- tol $= 0$ (finite convergence)
- z is a mixed-integer variable (mixed-integer stochastic linear programming)!