Multistage stochatic programming problems

The linear case

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Nested formulation

$$\inf_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[\inf_{x_2 \in \mathcal{X}_2(x_1,\xi_2)} f_2(x_2,\xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}[\inf_{x_T \in \mathcal{X}(x_{T-1},\xi_T)} f_T(x_T,\xi_T)]\right]\right]$$

- $\xi = (\xi_1, \ldots, \xi_T)$ is the stochastic process
- $f_t : \Re^{n_t} \times \Re^{d_t} \to \bar{\Re}$, $t = 1, \ldots, T$, are continuous functions
- $x_t \in \Re^{n_t}$, $t = 1, \ldots, T$, are the decision variables
- $\mathcal{X}_t : \Re^{n_{t-1}} \times \Re^{d_t} \rightrightarrows \Re^{n_t}$,$t = 1, \ldots, T$, are measurable, closed valued multifunctions

Nested formulation

$$\inf_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[\inf_{x_2 \in \mathcal{X}_2(x_1,\xi_2)} f_2(x_2,\xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}[\inf_{x_T \in \mathcal{X}(x_{T-1},\xi_T)} f_T(x_T,\xi_T)] \right]\right]$$

- $\xi = (\xi_1, \ldots, \xi_T)$ is the stochastic process
- $f_t(x_t, \xi_t) := c_t^\top x_t + i_{\Re_+^n}(x_t)$, $t = 1, \ldots, T$
- $x_t \in \Re^{n_t}$, $t = 1, \ldots, T$, are the decision variables
- $\mathcal{X}_t(x_{t-1}, \xi_t) := \{x_t \in \Re^{n_{t-1}} : B_t x_{t-1} + A_t x_t = b_t\}$, $t = 2, \ldots, T$

Multistage stochastic linear programs - T-SLP

Nested formulation

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}\left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \mathbb{E}\left[\cdots + \mathbb{E}[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T] \right]\right]$$

- Some elements of the data $\xi = (c_t, B_t, A_t, b_t)$ depend on uncertainties.

Multistage stochastic linear programs - T-SLP

Nested formulation

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}_{|\xi_1} \left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \mathbb{E}_{|\xi_{[2]}} \left[\cdots + \mathbb{E}_{|\xi_{[T-1]}} \left[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T \right] \right] \right].$$

- Some elements of the data $\xi = (c_t, B_t, A_t, b_t)$ depend on uncertainties.

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi_{[T]}) := \min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T$$

Dynamic programming formulation

- Stage $t = T$

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- At stages $t = 2, \ldots, T-1$

$$Q_t(x_{t-1}, \xi_{[t]}) := \min_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^\top x_t + \mathbb{E}_{|\xi_{[t]}} \left[Q_{t+1}(x_t, \xi_{[t+1]}) \right]$$

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi_{[T]}) := \min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T$$

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- Stage $t = 1$

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}\left[Q_2(x_1, \xi_2) \right]$$

Dynamic programming formulation

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathcal{Q}_2(x_1, \xi_{[1]})$$

Recourse function

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) := \mathbb{E}_{|\xi_{[t]}} \left[Q_{t+1}(x_t, \xi_{[t+1]}) \right]$$

Cost-to-go function

$$Q_t(x_{t-1}, \xi_{[t]}) := \min_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$$

Optimality conditions

- Let $f_t(x_t, \xi_t) = c_t^\top x_t + i_{\Re_+^{n_t}}(x_t)$
- $\bar{\pi}_t(\xi_{[t]}) \in \mathcal{D}(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_{[t]})$
- $\mathcal{D}(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_{[t]})$ is the set of Lagrange multipliers of

$$Q_t(x_{t-1}, \xi_{[t]}) := \min_{\substack{B_t x_{t-1} + A_t x_t = b_t \\ x_t \geq 0}} c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$$

Theorem

Under some assumptions (e.g. finitely many scenarios and polyhedral f_t). A feasible policy $\bar{x}_t(\xi_{[t]})$ is optimal iff there exists measurable $\bar{\pi}_t(\xi_{[t]})$, $t = 1, \ldots, T$, such that

$$0 \in \partial f_t(\bar{x}_t(\xi_{[t]}), \xi_t) - A_t^\top \bar{\pi}_t(\xi_{[t]}) - \mathbb{E}_{|\xi_{[t]}}[B_{t+1}^\top \bar{\pi}_{t+1}(\xi_{[t+1]})]$$

for a.e. $\xi_{[t]}$ and $t = 1, \ldots, T$.

Optimality conditions

- Let $f_t(x_t, \xi_t) = c_t^\top x_t + i_{\Re_+^{n_t}}(x_t)$
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Theorem

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$$0 \in \mathcal{N}_{\Re_+^{n_t}}(\bar{x}_t(\xi_{[t]})) + c_t - A_t^\top \bar{\pi}_t(\xi_{[t]}) - \mathbb{E}_{|\xi_{[t]}}[B_{t+1}^\top \bar{\pi}_{t+1}(\xi_{[t+1]})$$

for a.e. $\xi_{[t]}$ and $t = 1, \ldots, T$.

Scenario trees

- Assume that the stochastic process $\xi = (\xi_1, \ldots, \xi_T)$ has a finite number K of realizations
- Each realization (sequence) is called a scenario $\xi^i = (\xi_1^i, \ldots, \xi_T^i)$
- Each scenario $\xi^i = (\xi_1^i, \ldots, \xi_T^i)$ has a probability $p_i > 0$ associated
- The value of a given scenario ξ^i at stage t is denoted a node of the tree
- The set of all nodes at stage t is denoted by Ω_t
- The total number of scenario is $K = |\Omega_T|$
- We sometimes use the short hand ι to denote a node: $\iota \in \Omega_t$
- The ancestor of a node $\iota \in \Omega_t$ is $a(\iota) \in \Omega_{t-1}$
- The set of descendants (children) of a node $\iota \in \Omega_t$ is denoted by C_ι
- $\Omega_{t+1} = \cup_{\iota \in \Omega_t} C_\iota$, and $C_\iota \cap C_{\iota'} = \emptyset$ if $\iota \neq \iota'$
- $\mathcal{S}^{(\iota)}$ is the set of all scenarios passing through node ι
- The probability of node ι is $p^{(\iota)} := \mathbb{P}[\mathcal{S}^{(\iota)}]$
- Conditional probability $\rho_{a\iota} = \frac{p^{(\iota)}}{p^{(a)}}$ if $a = a(\iota)$
- The probability of reaching a node $\iota \in \Omega_t$ is $p^{(\iota)} = \rho_{\iota_1\iota_2}\rho_{\iota_2\iota_1} \ldots \rho_{\iota_{t-1}\iota_t} = \mathbb{P}[\mathcal{S}^{(\iota)}]$.

Nested formulation

Linear case

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}\left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \mathbb{E}\left[\cdots + \mathbb{E}[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T]\right]\right]$$

Linear case + scenario tree

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \sum_{\iota_2 \in \Omega_2} \rho_{1 \iota_2} \left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \sum_{\iota_3 \in \Omega_3} \rho_{\iota_2 \iota_3} \left[\cdots + \sum_{\iota_T \in \Omega} \rho_{\iota_{T-1} \iota_T} [\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T\right.\right.$$

Nested formulation

Linear case

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathbb{E}\left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \mathbb{E}\left[\cdots + \mathbb{E}[\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top x_T] \right]\right]$$

Linear case + scenario tree

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \sum_{\iota_2 \in \Omega_2} \rho_{1\iota_2} \left[\min_{\substack{B_2 x_1 + A_2 x_2 = b_2 \\ x_2 \geq 0}} c_2^\top x_2 + \sum_{\iota_3 \in \Omega_3} \rho_{\iota_2 \iota_3} \left[\cdots + \sum_{\iota_T \in \Omega} \rho_{\iota_{T-1}\iota_T} [\min_{\substack{B_T x_{T-1} + A_T x_T = b_T \\ x_T \geq 0}} c_T^\top \right.\right.$$

Ops! I forgot to name the nodes...

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \sum_{\iota_2 \in \Omega_2} \rho_{1\iota_2} \left[\min_{\substack{B_2^{\iota_2} x_1 + A_2^{\iota_2} x_2 = b_2^{\iota_2} \\ x_2 \geq 0}} c_2^{\iota_2 \top} x_2 + \sum_{\iota_3 \in \Omega_3} \rho_{\iota_2 \iota_3} \left[\cdots + \right.\right.$$

$$\left.\left.\left. \cdots + \sum_{\iota_T \in \Omega} \rho_{\iota_{T-1}\iota_T} \left[\min_{\substack{B_T^{\iota_T} x_{T-1} + A_T^{\iota_T} x_T = b_T^{\iota_T} \\ x_T \geq 0}} c_T^{\iota_T \top} x_T \right]\right]\right]$$

Equivalent deterministic

Denoting $\xi_t^{\iota_i} = (c_t^{\iota_i}, B_t^{\iota_i}, A_t^{\iota_i}, b_t^{\iota_i})$ we can rewrite the above problem as

$$\left\{\begin{array}{llllllll}
\min & c_1^\top x_1 & + & \sum_{\iota_2\in\Omega_2} p^{(\iota_2)} {c_2^{\iota_2}}^\top x_2^{\iota_2} & + & \sum_{\iota_3\in\Omega_3} p^{(\iota_3)} {c_3^{\iota_3}}^\top x_3^{\iota_3} & + \cdots + & \sum_{\iota_T\in\Omega} p^{(\iota_T)} {c_T^{\iota_T}}^\top x_T^{\iota_T} \\
\text{s.t.} & A_1 x_1 & & & & & & = b_1 \\
& B_2^{\iota_2} x_1 & + & A_2^{\iota_2} x_2^{\iota_2} & & & & = b_2^{\iota_2} \ \forall \iota_2 \in \Omega_2 \\
& & & B_3^{\iota_3} x_2^{a(\iota_3)} & + & A_3^{\iota_3} x_3^{\iota_3} & & = b_3^{\iota_3} \ \forall \iota_3 \in \Omega_3 \\
& & & \ddots & & \ddots & & \\
& & & B_T^{\iota_T} x_{T-1}^{a(\iota_T)} & + & A_T^{\iota_T} x_T^{\iota_T} & & = b_T^{\iota_T} \ \forall \iota_T \in \Omega_T
\end{array}\right.$$

This is a LP!

Equivalent deterministic

Denoting $\xi_t^{\iota i} = (c_t^{\iota i}, B_t^{\iota i}, A_t^{\iota i}, b_t^{\iota i})$ we can rewrite the above problem as

$$
\left\{
\begin{array}{llllllll}
\min & c_1^\top x_1 & + & \sum_{\iota_2 \in \Omega_2} p^{(\iota 2)} c_2^{\iota 2\,\top} x_2^{\iota 2} & + & \sum_{\iota_3 \in \Omega_3} p^{(\iota 3)} c_3^{\iota 3\,\top} x_3^{\iota 3} & + \cdots + & \sum_{\iota_T \in \Omega} p^{(\iota T)} c_T^{\iota T\,\top} x_T^{\iota T} \\
\text{s.t.} & A_1 x_1 & & & & & & = b_1 \\
 & B_2^{\iota 2} x_1 & + & A_2^{\iota 2} x_2^{\iota 2} & & & & = b_2^{\iota 2} \ \forall \iota_2 \in \Omega_2 \\
 & & & B_3^{\iota 3} x_2^{a(\iota 3)} & + & A_3^{\iota 3} x_3^{\iota 3} & & = b_3^{\iota 3} \ \forall \iota_3 \in \Omega_3 \\
 & & & \ddots & & \ddots & & \\
 & & & B_T^{\iota T} x_{T-1}^{a(\iota T)} & + & A_T^{\iota T} x_T^{\iota T} & & = b_T^{\iota T} \ \forall \iota_T \in \Omega_T
\end{array}
\right.
$$

This is a LP!

Example:

- Suppose $T = 12$, each node $\iota_t \in \Omega_t$ has 4 children nodes

This gives $4^{11} = 4,194,304$ scenarios.

- Suppose each $x_t \in \Re^{100}$, $t = 1, \ldots, 12$

The number of variables of the above problem is approximately 4.59×10^8!

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi^{\iota}_{[T]}) := \min_{\substack{B^{\iota}_T x^{a(\iota)}_{T-1} + A^{\iota}_T x_T = b^{\iota}_T \\ x_T \geq 0}} {c^{\iota}_T}^\top x_T$$

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi^\iota_{[T]}) := \min_{\substack{B^\iota_T x^{a(\iota)}_{T-1} + A^\iota_T x_T = b^\iota_T \\ x_T \geq 0}} {c^\iota_T}^\top x_T$$

- At stages $t = 2, \dots, T-1$

$$Q_t(x_{t-1}, \xi^\iota_{[t]}) := \min_{\substack{B^\iota_t x^{a(\iota)}_{t-1} + A^\iota_t x_t = b^\iota_t \\ x_t \geq 0}} {c^\iota_t}^\top x_t + \sum_{j \in C_\iota} p^{(j)} \left[Q_{t+1}(x_t, \xi^j_{[t+1]}) \right]$$

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi^\iota_{[T]}) := \min_{\substack{B^\iota_T x^{a(\iota)}_{T-1} + A^\iota_T x_T = b^\iota_T \\ x_T \geq 0}} c^{\iota\ \top}_T x_T$$

- At stages $t = 2, \ldots, T-1$

$$Q_t(x_{t-1}, \xi^\iota_{[t]}) := \min_{\substack{B^\iota_t x^{a(\iota)}_{t-1} + A^\iota_t x_t = b^\iota_t \\ x_t \geq 0}} c^{\iota\ \top}_t x_t + \sum_{j \in C_\iota} p^{(j)} \left[Q_{t+1}(x_t, \xi^j_{[t+1]}) \right]$$

- Stage $t = 1$

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \sum_{\iota \in C_1} p^{(\iota)} \left[Q_2(x_1, \xi^\iota_2) \right]$$

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi^{\iota}_{[T]}) := \min_{\substack{B^{\iota}_T x^{a(\iota)}_{T-1} + A^{\iota}_T x_T = b^{\iota}_T \\ x_T \geq 0}} {c^{\iota}_T}^{\top} x_T$$

- At stages $t = 2, \ldots, T-1$

$$\underline{Q}_t(x_{t-1}, \xi^{\iota}_{[t]}) := \min_{\substack{B^{\iota}_t x^{a(\iota)}_{t-1} + A^{\iota}_t x_t = b^{\iota}_t \\ x_t \geq 0}} {c^{\iota}_t}^{\top} x_t + \sum_{j \in C_{\iota}} p^{(j)} \left[\underline{Q}_{t+1}(x_t, \xi^{j}_{[t+1]}) \right]$$

- Stage $t = 1$

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^{\top} x_1 + \sum_{\iota \in C_1} p^{(\iota)} \left[\underline{Q}_2(x_1, \xi^{\iota}_2) \right]$$

Dynamic programming formulation

- Stage $t = T$

$$Q_T(x_{T-1}, \xi^{\iota}_{[T]}) := \min_{\substack{B^{\iota}_T x^{a(\iota)}_{T-1} + A^{\iota}_T x_T = b^{\iota}_T \\ x_T \geq 0}} {c^{\iota}_T}^{\top} x_T$$

- At stages $t = 2, \dots, T-1$

$$\underline{Q}_t(x_{t-1}, \xi^{\iota}_{[t]}) := \min_{\substack{B^{\iota}_t x^{a(\iota)}_{t-1} + A^{\iota}_t x_t = b^{\iota}_t \\ x_t \geq 0}} {c^{\iota}_t}^{\top} x_t + \check{\mathcal{Q}}_{t+1}(x_t, \xi^{\iota}_{[t]})$$

$$\check{\mathcal{Q}}_{t+1}(x_t, \xi^{\iota}_{[t]}) := \sum_{j \in C_{\iota}} p^{(j)} \left[\underline{Q_{t+1}}(x_t, \xi^{j}_{[t+1]}) \right]$$

- Stage $t = 1$

$$\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^{\top} x_1 + \check{\mathcal{Q}}_2(x_2, \xi_{[1]})$$

Cutting-plane approximation

Assumptions

- The set of nodes Ω_t has finitely many elements
- the problem has recourse relatively complete

The last hypotheses is made only for sake of simplicity!

Cutting-plane approximation

- Stage $t = T$

$$Q_T(x_{T-1}^k, \xi_{[T]}) := \min_{\substack{B_T x_{T-1}^k + A_T x_T = b_T \\ x_T \geq 0}} c_T{}^\top x_T$$

- At stages $t = 2, \ldots, T-1$

$$\underline{Q_t}(x_{t-1}^k, \xi_{[t]}) := \begin{cases} \min\limits_{x_t \geq 0, r_{t+1}} & c_t{}^\top x_t + r_{t+1} \\ \text{s.t.} & B_t x_{t-1}^k + A_t x_t = b_t \\ & r_{t+1} \geq \alpha_{t+1}^j + \beta_{t+1}^j x_t \quad j = 1, \ldots, k \end{cases}$$

- Stage $t = 1$

$$\underline{z}^k := \begin{cases} \min\limits_{x_1 \geq 0, r_2} & c_1^\top x_1 + r_2 \\ \text{s.t.} & A_1 x_1 = b_1 \\ & r_2 \geq \alpha_2^j + \beta_2^j x_1 \quad j = 1, \ldots, k \end{cases}$$

Cutting-plane approximation

- At stages $t = 2, \dots, T-1$

$$\underline{Q}_t(x_{t-1}^k, \xi_{[t]}) := \begin{cases} \min\limits_{x_t \geq 0, r_{t+1}} & {c_t}^\top x_t + r_{t+1} \\ \text{s.t.} & B_t x_{t-1}^k + A_t x_t = b_t \qquad\qquad (\pi_t) \\ & r_{t+1} \geq \alpha_{t+1}^j + \beta_{t+1}^j x_t \quad j = 1, \dots, k \quad (\rho_j) \end{cases}$$

- Cuts $(t = T)$

$$\alpha_T^k := \mathbb{E}_{|\xi_{T-1}}[b_T^\top \pi_T^k] \quad \text{and} \quad \beta_T^k := -\mathbb{E}_{|\xi_{T-1}}[B_T^\top \pi_T^k]$$

- Cuts $(t = T-1, \dots, 2)$

$$\alpha_t^k := \mathbb{E}_{|\xi_{t-1}}[b_t^\top \pi_t^k + \sum_{j=1}^{k} \alpha_{t+1}^j \rho_k^j] \quad \text{and} \quad \beta_t^k := -\mathbb{E}_{|\xi_{t-1}}[B_t^\top \pi_t^k]$$

Nested decomposition - (nested L-shaped method)

- J.R. Birge (1985).

It has two main steps:

- Forward that goes from $t = 1$ up to $t = T$ solving subproblems to define policy $x_t^k(\xi_t)$.
  - In this step an upper bound $\overline{z}^k$ for the optimal value is determined.

- Backward that comes from $t = T$ up to $t = 1$ solving subproblems to compute linearizations that improve the cutting-plane approximation.
  - In this step a lower bound $\underline{z}^k$ is obtained.

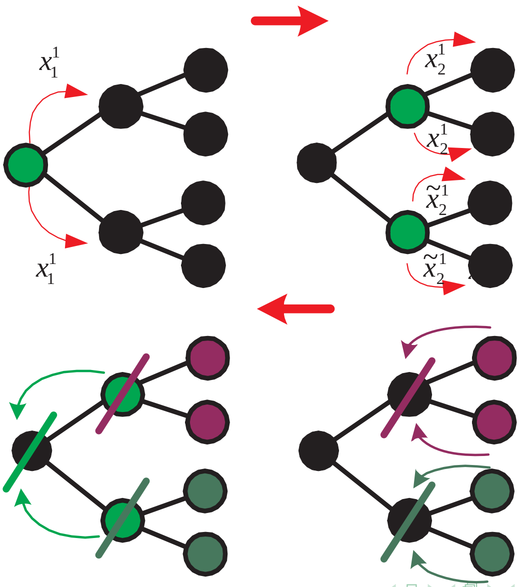
Stopping test

- The Nested decomposition stops when

$$\overline{z}^k - \underline{z}^k \leq \texttt{Tol}.$$

- In this case x_1^k is a `Tol`-solution to the T-SLP.

Forward and Backward steps



Algorithm - Nested decomposition

stages $t = 2, \dots, T-1$

$$\underline{Q}_t(x_{t-1}^k, \xi_{[t]}) := \begin{cases} \min\limits_{x_t \geq 0, r_{t+1}} & c_t^\top x_t + r_{t+1} \\ \text{s.t.} & B_t x_{t-1}^k + A_t x_t = b_t & & (\pi_t) \\ & r_{t+1} \geq \alpha_{t+1}^j + \beta_{t+1}^j x_t & j = 1, \dots, k & (\rho_j) \end{cases}$$

- **Step 0: inicialization**. Define $k = 1$ and add the constraint $r_t = 0$ in all LPs $\underline{Q}_t$, $t = 2, \dots, T-1$. Compute $\underline{z}^1$ and let its solution be x_1^1.

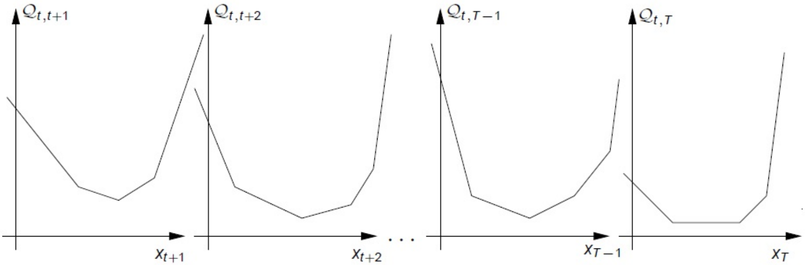
- **Step 1: forward**. For t=2,...,T, solve the LP $\underline{Q}_t$ to obtain $x_t^k := x_t^k(\xi_{[t]})$. Define $\bar{z}^k := \mathbb{E}[\sum_{t=1}^T c_t^\top x_t^k]$.

- **Step 2: backward**. Compute α_T^k and β_T^k. Set $t = T$. Loop:
  - While $t > 2$
  - $t \leftarrow t - 1$
  - solve the LP $\underline{Q}_t(x_{t-1}^k, \xi_{[t]})$
  - Compute α_t^k and β_t^k

  Compute $\underline{z}^k$ and let its solution be x_1^{k+1}.

- **Step 3: Stopping test**. If $\bar{z}^k - \underline{z}^k \leq \epsilon$, stop. Otherwise set $k \leftarrow k + 1$ and go back to Step 1.

Nested decomposition - iterative process

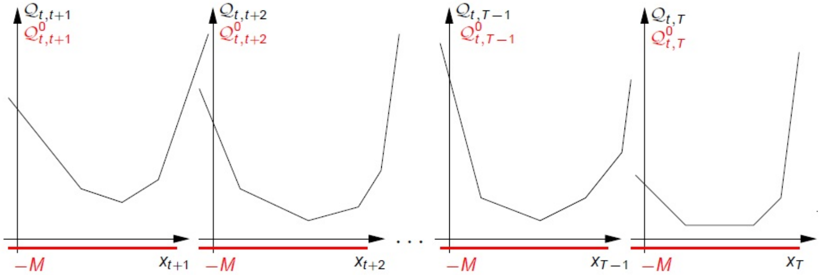


- $\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \mathcal{Q}_2(x_2, \xi_{[1]})$

- $\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) = \mathbb{E}_{|\xi_{[t]}}[Q_{t+1}(x_t, \xi_{[t+1]})]$ for $t = 1, \ldots, T-1$, e
  $\mathcal{Q}_{T+1}(x_T, \xi_{[T]}) = 0$

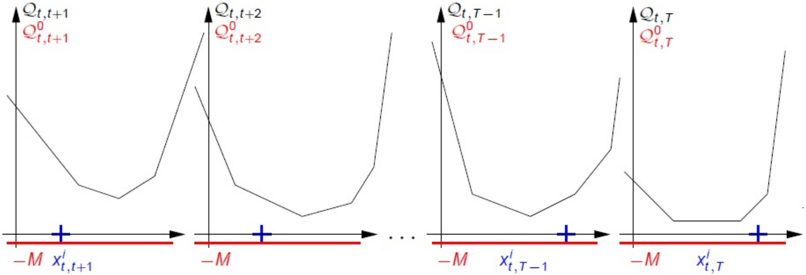
- $Q_t(x_{t-1}, \xi_{[t]}) = \min\ c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$ s.t. $B_t x_{t-1} + \underset{x_t \geq 0}{A_t x_t} = b_t$.

Nested decomposition - iterative process



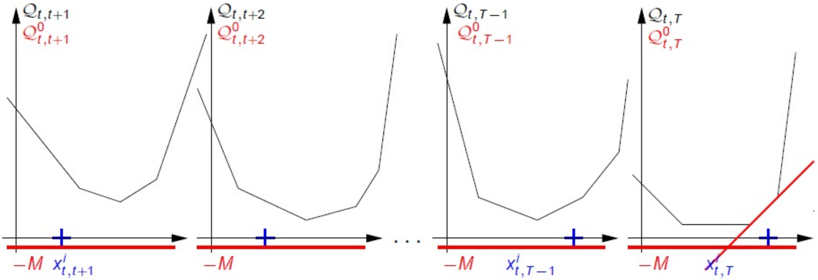
- $\min_{\substack{A_1 x_1 = b_1 \\ x_1 \geq 0}} c_1^\top x_1 + \check{\mathcal{Q}}_2(x_2, \xi_{[1]})$
- $\check{\mathcal{Q}}_{t+1}(x_t, \xi_{[t]}) = \mathbb{E}_{|\xi_{[t]}}[\underline{Q_{t+1}}(x_t, \xi_{[t+1]})]$ for $t = 1, \ldots, T-1$, e $\check{\mathcal{Q}}_{T+1}(x_T, \xi_{[T]}) = 0$
- $\underline{Q_t}(x_{t-1}, \xi_{[t]}) = \min_{x_t \geq 0} c_t^\top x_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$ s.t. $B_t x_{t-1} + A_t x_t = b_t$.

Nested decomposition - iterative process



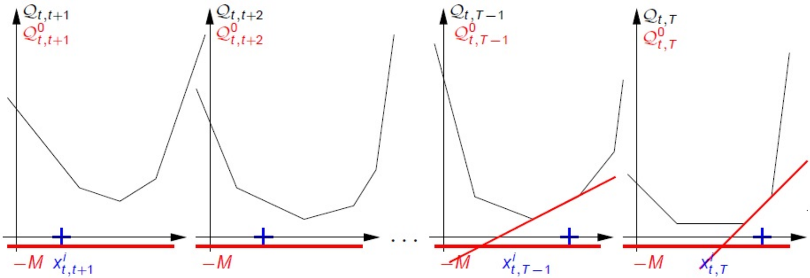
Forward step

Nested decomposition - iterative process



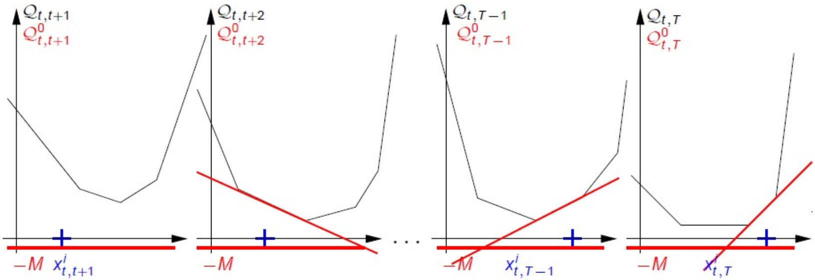
Backward step

Nested decomposition - iterative process



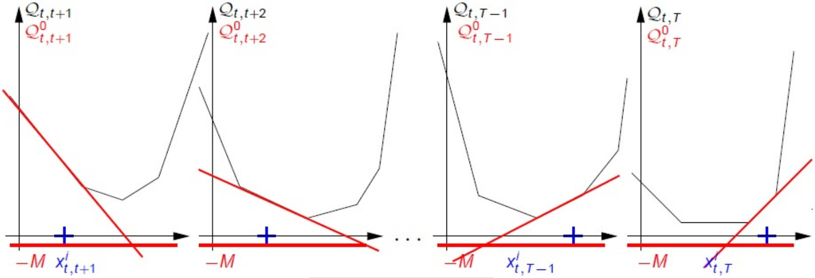
Backward step

Nested decomposition - iterative process



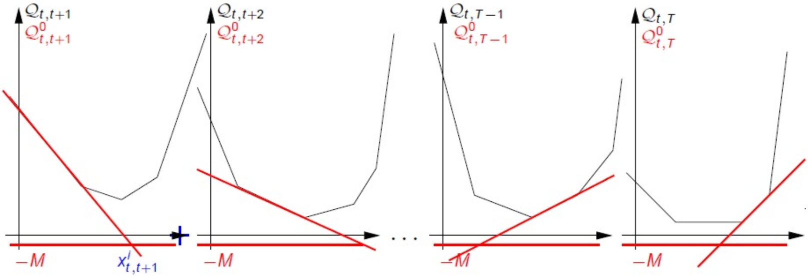
Backward step

Nested decomposition - iterative process



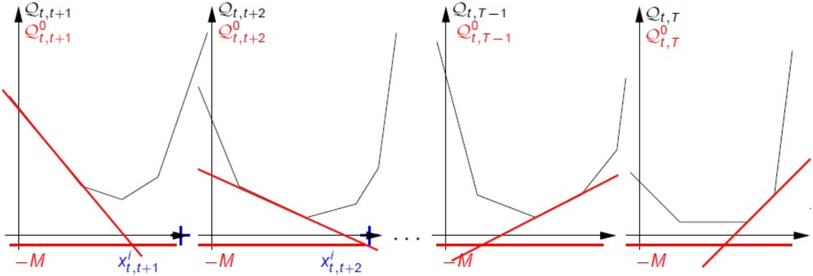
Backward step

Nested decomposition - iterative process



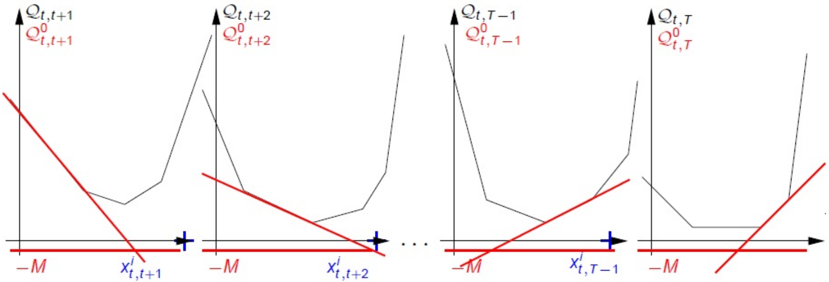
Forward step

Nested decomposition - iterative process



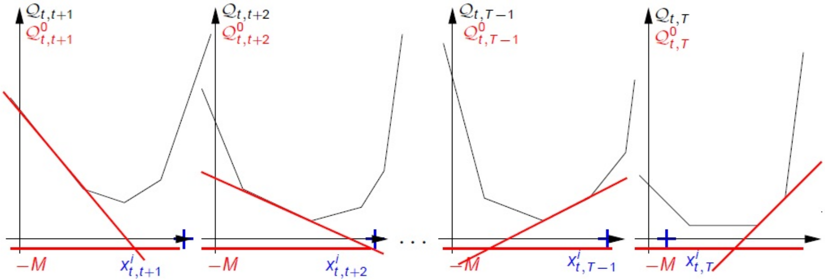
Forward step

Nested decomposition - iterative process



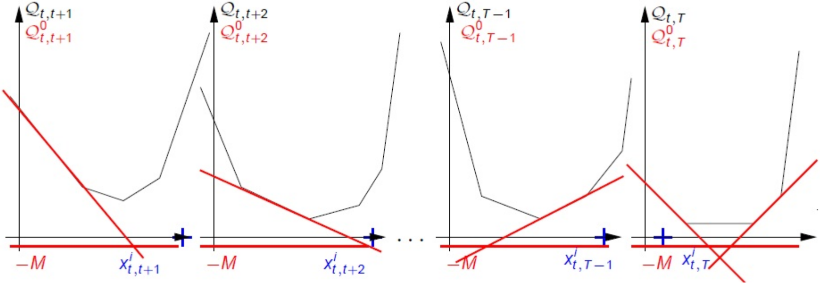
Forward step

Nested decomposition - iterative process



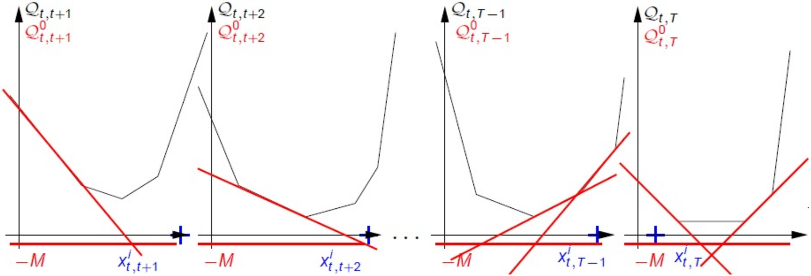
Forward step

Nested decomposition - iterative process



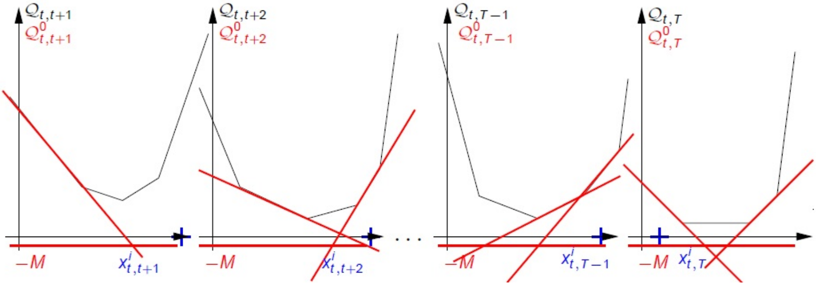
Backward step

Nested decomposition - iterative process



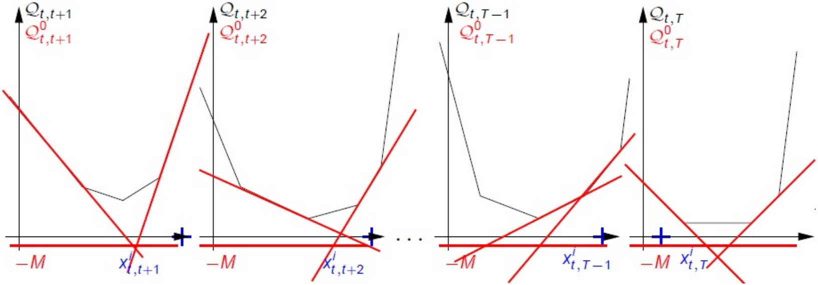
Backward step

Nested decomposition - iterative process



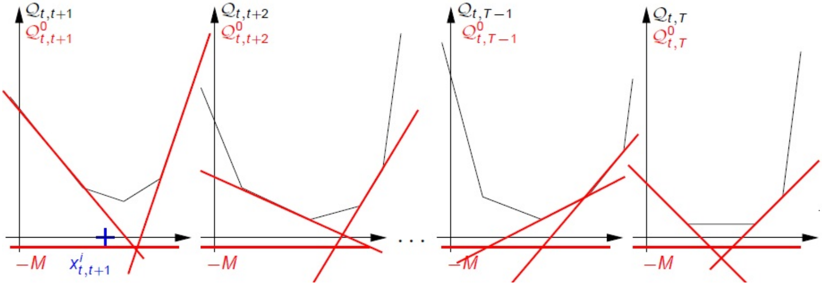
Backward step

Nested decomposition - iterative process



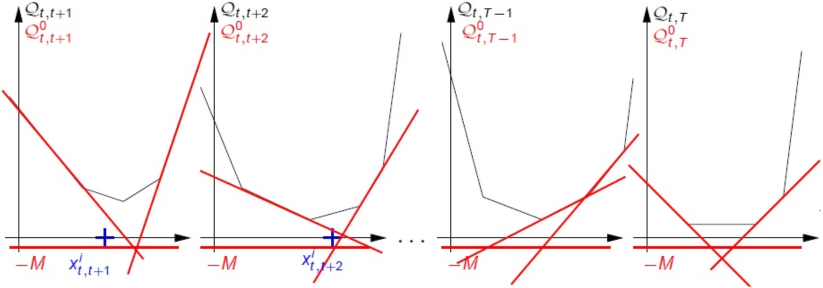
Backward step

Nested decomposition - iterative process



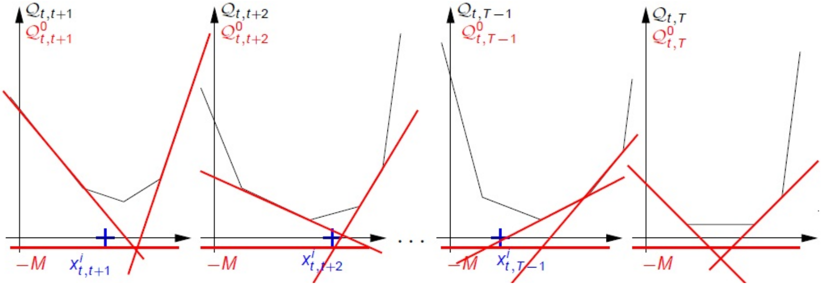
Forward step

Nested decomposition - iterative process



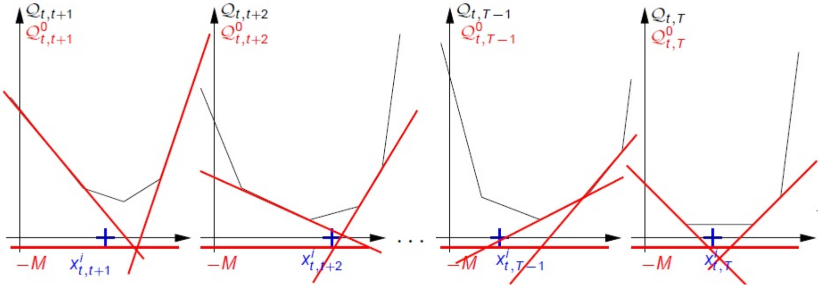
Forward step

Nested decomposition - iterative process



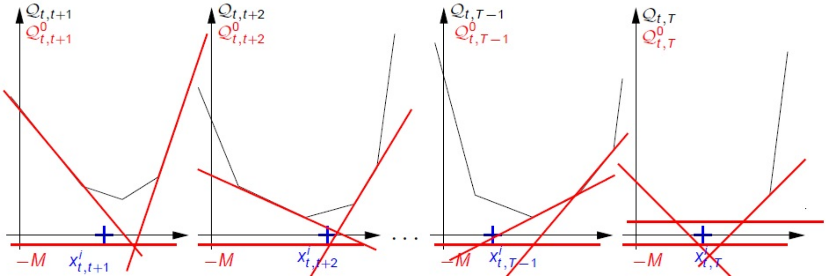
Forward step

Nested decomposition - iterative process



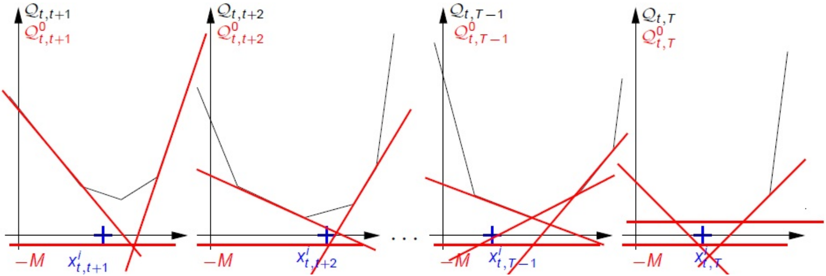
Forward step

Nested decomposition - iterative process



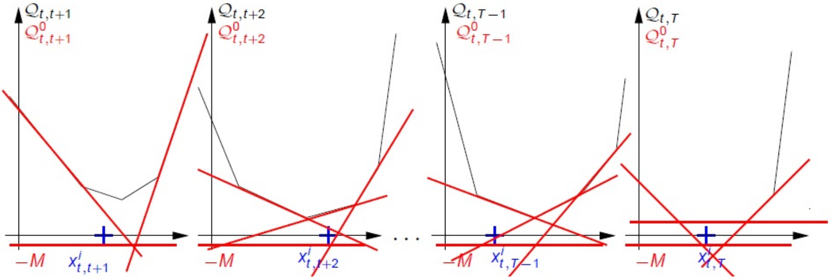
Backward step

Nested decomposition - iterative process



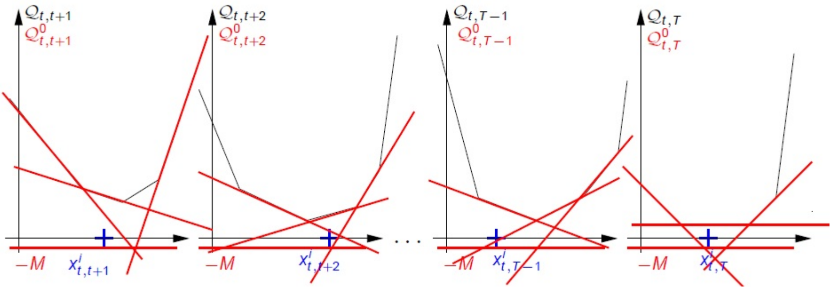
Backward step

Nested decomposition - iterative process



Backward step

Nested decomposition - iterative process



Backward step

- Figures by Vincent Guigues.

Convergence analysis

Assumptions

- The set of nodes Ω_t has finitely many elements, $t = 1, \ldots, T$
- the problem has recourse relatively complete (for simplicity, only)
- the feasible set, in each stage $t = 1, \ldots, T$, is compact

Lemma

$\check{\mathcal{Q}}_t^k(x_{t-1}, \xi_{[t-1]}) \leq \mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}) \quad \forall\, x_{t-1} \quad \textit{and} \quad \forall t = 2, \ldots, T$

Theorem

The Nested Decomposition converges finitely to an optimal solution of the considered T-SLP.