# Risk Measures

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# Functions to the Extended Real Line

Given $f : \mathcal{M} \to \overline{\mathbb{R}}$. The epigraph of $f$ is:

$$\operatorname{epf}(f) = \{(x, t) \in \mathcal{M} \times \mathbb{R} : f(x) \leq t\}$$

The effective domain is $\operatorname{dom}(f) = \{x \in \mathcal{M} : f(x) < +\infty\}$
Note that $\operatorname{dom}(f) = \operatorname{proj}_{\mathcal{M}}(\operatorname{epf}(f))$
Given a real number $l$ we define the sub-level $[f \leq l]$ by

$$[f \leq l] = \{x \in \mathcal{M} : f(x) \leq l\}$$

The function is called <u>proper</u> if $f \not\equiv +\infty$ and $f > -\infty$.

# Functions to the Extended Real Line

Examples

## Supremum Function

Given a family $f_\lambda : \mathcal{M} \to \overline{\mathbb{R}}$, for $\lambda \in \Lambda$. The supremum function $f(x) = \sup_{\lambda \in \Lambda} f_\lambda(x) \in \overline{\mathbb{R}}$ is well defined. Also $\mathrm{epf}(f) = \cap_{\lambda \in \Lambda} \mathrm{epf}(f_\lambda)$

## Sum of Functions

Given a sequence of functions $f_i : \mathcal{M} \to \overline{\mathbb{R}}$, for $i = 1, 2, \ldots, m$. Whenever $(\cup_{i=1}^m \{x : f_i(x) = +\infty\}) \cap (\cup_{i=1}^m \{x : f_i(x) = -\infty\}) = \phi$, the sum function $f(x) = \sum_{i=1}^m f_i(x)$ is well defined. Also we have that $\mathrm{dom}(f) = \cap_{i=1}^m \mathrm{dom}(f_i)$.

## Indicator Function

Given any subset $D \subset \mathcal{M}$, the indicator function is defined by

$$\delta_D(x) = \begin{cases} 0 & \text{if } x \in D \\ +\infty & \text{otherwise} \end{cases}$$

# Lower Semicontinuous Functions

A function $f : \mathcal{M} \to \overline{\mathbb{R}}$ is lower semicontinuous (l.s.c.) if its epigraph $\mathrm{epi}(f)$ is closed in $\mathcal{M} \times \mathbb{R}$.

1. A function $f$ is l.s.c. if and only if the sub-level set $[f \leq l]$ is closed for all $l \in \mathbb{R}$.
2. If $\{f_\lambda\}_{\lambda \in \Lambda}$ is a family of l.s.c. functions, then $\sup_{\lambda \in \Lambda} f_\lambda$ is also l.s.c.

# Convex Functions

"$f$ is convex if $epf(f) \subset \mathcal{M} \times \mathbb{R}$ is a convex set."

Equivalently: $f$ is convex if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y), \quad \forall x, y \in \operatorname{dom}(f), \forall t \in [0,1]$$

- If $f$ is convex $\Rightarrow \operatorname{dom}(f)$ is convex.
- If $f$ is convex $\Rightarrow \{x : f(x) \leq M\}$ is convex, for all $M \in \mathbb{R}$

## Fenchel Conjugate

For a normed space $\mathcal{M}$, there exists its dual space $\mathcal{M}^*$ and a duality product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M}^* \to \mathbb{R}$. Considering these elements, for a function $f : \mathcal{M} \to \overline{\mathbb{R}}$, we define its Fenchel conjugate $f^* : \mathcal{M}^* \to \overline{\mathbb{R}}$ by

$$f^*(x^*) = \sup_{x \in \mathcal{M}} \langle x, x^* \rangle - f(x)$$

- $f^*$ is convex l.s.c.
- If $g \leq f$, then $f^* \leq g^*$.

# Fenchel Conjugate

### Proposition

Given a function $f : \mathcal{M} \to \overline{\mathbb{R}}$, we have

- If $f$ is proper and there exist $x^* \in \mathcal{M}^*$ and $\alpha \in \mathbb{R}$ such that $f(x) \geq \langle x, x^* \rangle - \alpha$ for all $x$, then $f^*$ is proper.
- If $f$ is convex proper l.s.c., then $f^*$ is proper.

### Proposition

If $f$ is convex, then $x^* \in \partial f(x)$ if and only if $f(x) + f^*(x^*) = \langle x, x^* \rangle$.

# Fenchel Conjugate

### Lemma
For any function $f$ we have that $f \geq f^{**}$.

### Lemma
If $f$ is a proper convex l.s.c function then

$$f(x) = \sup_{L \leq f,\, L \text{ affine}} L(x)$$

### Theorem
If $f$ is a proper convex l.s.c function then

$$f = f^{**}$$

**Remark.** For any function $f$, if $g$ is proper convex l.s.c. and $g \leq f$, then $g \leq f^{**} \leq f$.

# Coherent Risk Measures

For a risk measure $\rho : \mathcal{M} \to \overline{\mathbb{R}}$ with conjugate $\rho^* : \mathcal{M}^* \to \overline{\mathbb{R}}$ we have that

- Translation invariance of $\rho$ implies $\mathbb{E}[Z^*] = 1$, for all $Z^* \in \mathrm{dom}(\rho^*)$.
- If is $\rho$ monotone, then $Z^* \geq 0$, for all $Z^* \in \mathrm{dom}(\rho^*)$.
- If is $\rho$ positively homogeneous, then $\rho^* = \delta_{\mathrm{dom}(\rho^*)}$.

# Coherent Risk Measures

### Proposition

A proper risk measure $\rho$ is coherent l.s.c. if and only if there exists a convex closed $\mathcal{U} \subset \{Z^* \geq 0 : \mathbb{E}[Z^*] = 1\}$ such that

$$\rho(Z) = \sup_{Z^* \in \mathcal{U}} \mathbb{E}[Z^* Z]$$

On these cases we have that

$$\partial\rho(Z) = \operatorname{argmax}_{Z^* \in \mathcal{U}} \mathbb{E}[Z^* Z] \subset \partial\rho(0) = \mathcal{U} = \operatorname{dom}(\rho^*)$$

**Example.** $\mathcal{U} = \{\chi_\Omega\}$ and $\mathcal{U} = \{Z^* \geq 0 : \mathbb{E}[Z^*] = 1\}$

# Coherent Risk Measures

### Proposition

For $0 < \alpha < 1$ we have that

$$AVaR_\alpha(Z) = \max_{Z^* \in \mathcal{U}} \mathbb{E}[Z^* Z]$$

where $\mathcal{U} = \{Z^* \geq 0 : Z^* \leq \frac{1}{\alpha}, \mathbb{E}[Z^*] = 1\}$.
Moreover

$$\partial AVaR_\alpha(Z) = \left\{ Z^* : \begin{array}{ll} Z^* = \frac{1}{\alpha} & \text{if } Z > v \\ Z^* \in [0, \frac{1}{\alpha}] & \text{if } Z = v \\ Z^* = 0 & \text{if } Z < v \end{array}, \mathbb{E}[Z^*] = 1 \right\}$$

where $v$ is a minimizer for the optimization problem that defines $AVaR_\alpha(Z)$.

**Proof.** We already know that $\mathrm{dom}(AVaR^*) = \mathcal{U} \subset \{Z^* \geq 0 : \mathbb{E}[Z^*] = 1\}$. On the other hand, from the definition we have that

$$\begin{aligned}
AVaR^*(Z^*) &= \sup_Z \{\mathbb{E}[Z^* Z] - AVaR_\alpha(Z)\} \\
&= \sup_Z \{\mathbb{E}[Z^* Z] - \min_u \{u + \frac{1}{\alpha}\mathbb{E}[[Z-u]^+]\}\} \\
&= \sup_{Z,u} \{\mathbb{E}[Z^* Z] - u - \frac{1}{\alpha}\mathbb{E}[[Z-u]^+]\} \\
&= \sup_{Z,u} \{\mathbb{E}[Z^*(Z-u)] - \frac{1}{\alpha}\mathbb{E}[[Z-u]^+]\} \\
&= \sup_{Z,u} \{\mathbb{E}[(Z^* - \frac{1}{\alpha})[Z-u]^+] - \mathbb{E}[Z^*[Z-u]^-]\}
\end{aligned}$$

Now, we will prove that if $Z^* \in \mathcal{U}$, then $Z^* \leq 1/\alpha$. For that, assume that the set $A := [Z^* > 1/\alpha]$ has positive probability. Then for each natural number $n$ we can consider the random variable $Z_n := n\chi_A$. Then, plugging $Z_n$ and $u = 0$ in the last equality above, we have that

$$\begin{aligned}
AVaR^*(Z^*) &= \sup_{Z,u} \{\mathbb{E}[(Z^* - \frac{1}{\alpha})[Z-u]^+] - \mathbb{E}[Z^*[Z-u]^-]\} \\
&\geq \mathbb{E}[(Z^* - \frac{1}{\alpha})[Z_n]^+] = \mathbb{E}[(Z^* - \frac{1}{\alpha})n\chi_A] = n\mathbb{E}[(Z^* - \frac{1}{\alpha})\chi_A] > 0
\end{aligned}$$

then, as $n \to +\infty$ we have that $AVaR^*(Z^*) \to \infty$, which contradicts the fact that $Z^* \in \mathcal{U}$.
Thus $Z^* \leq 1/\alpha$. Also, taking $Z^* \in \{Z^* \geq 0, \mathbb{E}[Z^*] = 1, Z^* \leq 1/\alpha\}$, we have that
$Z^*(Z-u) \leq \frac{1}{\alpha}[Z-u]^+, \forall Z, u$, and so
$AVaR^*(Z^*) = \sup_{Z,u}\{\mathbb{E}[Z^*(Z-u)] - \frac{1}{\alpha}\mathbb{E}[[Z-u]^+]\} \leq 0 < \infty$, which implies that $Z^* \in \mathcal{U}$.
In this way, we have proven that $\mathcal{U} = \{Z^* \geq 0, \mathbb{E}[Z^*] = 1, Z^* \leq 1/\alpha\}$.
Now, we will characterize $\partial AVaR_\alpha(Z)$. For this, note that $Z^* \in \partial AVaR_\alpha(Z)$ iff $AVaR_\alpha(Z) = \mathbb{E}[Z^* Z]$ and $Z^* \in \mathcal{U}$. Let $v$ a minimizer of

$AVaR_\alpha(Z) = \min_u u + (1/\alpha)\mathbb{E}[[Z-u]^+]$. Then $Z^* \in \partial AVaR_\alpha(Z)$ iff $Z^* \in \mathcal{U}$ and $\mathbb{E}[Z^* Z] = v + (1/\alpha)\mathbb{E}[[Z-v]^+]$. This is equivalent to

$$\begin{aligned} 0 &= \mathbb{E}[Z^* Z] - v - (1/\alpha)\mathbb{E}[[Z-v]^+] \\ &= \mathbb{E}[Z^*(Z-v) - (1/\alpha)[Z-v]^+] \end{aligned}$$

Since $Z^*(Z-v) - (1/\alpha)[Z-v]^+ \leq 0$, we have that $Z^*(Z-v) - (1/\alpha)[Z-v]^+ = 0$. From this equality,

1. For $\omega \in [Z > v]$, we have $(Z_\omega - v) = [Z_\omega - v]^+ > 0$, so, since $Z^*_\omega(Z_\omega - v) - (1/\alpha)[Z_\omega - v]^+ = 0$, we have that $Z^*_\omega = 1/\alpha$.
2. For $\omega \in [Z < v]$, we have $(Z_\omega - v) < 0$ and $[Z_\omega - v]^+ = 0$, so, we have that $Z^*_\omega(Z_\omega - v) = 0$, thus $Z^*_\omega = 0$.

This shows that

$$\partial AVaR_\alpha(Z) \subset \left\{ Z^* : \begin{array}{ll} Z^* = \frac{1}{\alpha} & \text{if } Z > v \\ Z^* \in [0, \frac{1}{\alpha}] & \text{if } Z = v \\ Z^* = 0 & \text{if } Z < v \end{array}, \mathbb{E}[Z^*] = 1 \right\}$$

Proving the other inclusion is straightforward.