Risk Measures

Juan Pablo Luna.

BAS Lecture 21, May 21, 2016

SVAN 2016

Given a random variable Z.

1. For $0 < \alpha < 1$ we have that
   $\mathbb{P}[Z > VaR_\alpha(Z)] \leq \alpha \leq \mathbb{P}[Z \geq VaR_\alpha(Z)]$
2. For $\mathbb{P}[Z > VaR_\alpha(Z)] \leq \beta < \mathbb{P}[Z \geq VaR_\alpha(Z)]$ we have that
   $VaR_\alpha(Z) = VaR_\beta(Z)$.

So, if Z has an atom at $VaR_\alpha(Z)$, then VaR is blind to some changes at risk level.

Coherent Risk Measures

A risk measure

$$\rho : \mathcal{M} \to \overline{\mathbb{R}}$$
$$Z \to \rho(Z)$$

is said to be coherent if it satisfies

- **Translation Equivariance:** $\rho(Z + a) = \rho(Z) + a, \quad \forall Z, \forall a \in \mathbb{R}$
- **Monotonicity:** $\rho(Z) \leq \rho(Y), \quad \text{if } Z \leq Y$
- **Subadditivity:** $\rho(Y + Z) \leq \rho(Y) + \rho(Z), \quad \forall Y, Z$
- **Positive Homogeneity:** $\rho(aZ) = a\rho(Z), \quad \forall a \geq 0, \forall Z.$

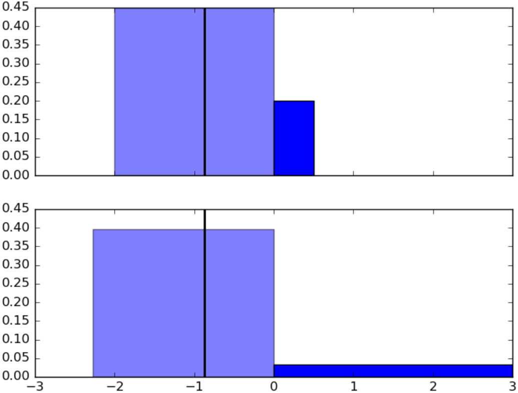
Example. VaR is not coherent.

Remark

$$\begin{bmatrix} \textbf{Subadditivity} \\ \textbf{Positive Homogeneity} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \textbf{Convexity} \\ \textbf{Positive Homogeneity} \end{bmatrix}$$

AVaR

Two distributions with $VaR_{0.1} = 0$.



VaR is myopic to values worse than itself.

AVaR

Consider a continuous random variable $Z \in L^1$ and $0 < \alpha < 1$. We define the Average Value at Risk (at level α) by

$$\begin{aligned} AVaR_\alpha(Z) &= \mathbb{E}[Z|Z > VaR_\alpha(Z)] \\ &= \mathbb{E}[Z|Z \geq VaR_\alpha(Z)] \\ &= \frac{\mathbb{E}[Z\chi_{[Z \geq VaR_\alpha(Z)]}]}{\mathbb{P}[Z \geq VaR_\alpha(Z)]} \\ &= \frac{1}{\alpha}\mathbb{E}[Z\chi_{[Z \geq VaR_\alpha(Z)]}] \end{aligned}$$

AVaR as optimal value

$$AVaR_\alpha(Z) = VaR_\alpha(Z) + \frac{1}{\alpha}\mathbb{E}\left[[Z - VaR_\alpha(Z)]^+\right]$$

Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(u) = u + \frac{1}{\alpha}\mathbb{E}\left[[Z - u]^+\right]$$

- g is finite.
- g is convex.
- $AVaR_\alpha(Z) = g(VaR_\alpha(Z))$

AVaR as optimal value

Proposition

The function g is differentiable and $g'(VaR_\alpha(Z))$. Thus, $VaR_\alpha(Z)$ is a minimizer of

$$AVaR_\alpha(Z) = \min_u u + \frac{1}{\alpha}\mathbb{E}\left[[Z-u]^+\right]$$

For computing AVaR, instead of computing VaR and expected value, you can solve an optimization problem.

AVaR for general distributions

Proposition

Given a random variable $Z \in L^1$ and $0 < \alpha < 1$, we have that

1. The function $g(u) = u + \frac{1}{\alpha}\mathbb{E}\left[[Z-u]^+\right]$ is finite and convex.
2. The optimization problem

$$\min_u u + \frac{1}{\alpha}\mathbb{E}\left[[Z-u]^+\right]$$

   always has solution.
3. $VaR_\alpha(Z)$ is a minimizer of this optimization problem.

AVaR for general distributions

Definition

For a general random variable $Z \in L^1$ and $0 < \alpha < 1$, we define

$$AVaR_\alpha(Z) := \min_u u + \frac{1}{\alpha}\mathbb{E}\left[[Z-u]^+\right]$$

- This definition generalizes $AVaR_\alpha(Z) = \mathbb{E}[Z|Z > VaR_\alpha(Z)]$, for continuous random variables.
- The definition as optimal value does not gives any clue of its meaning for general distribution. However it has appealing properties.
- $AVaR_\alpha(Z) \geq VaR_\alpha(Z)$.

AVaR for general distributions

Proposition

AVaR is a coherent risk measure.

Proposition

Given $Z \in L^1$.

1. If $0 < \alpha \leq \beta < 1$, then $AVaR_\beta(Z) \leq AVaR_\alpha(Z)$.
2. $\lim_{\alpha \uparrow 1} AVaR_\alpha(Z) = \mathbb{E}[Z] = \inf_u u + \mathbb{E}\left[[Z-u]^+\right]$.
3. The function $\alpha \to AVaR_\alpha(Z)$ is left continuous.

The level α can be regarded as being a risk tolerance level.