Risk Measures

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Risk

On stochastic programming, frequently we face situations where:

1. There a future event we cannot forecast
2. Our situation, when the uncertain event takes place, depends on the decision we make today.

So, it is important make good decisions today to avoid bad situations in the future.

Risk

1. We model the uncertain future outcomes by a random vector $\xi : \Omega \to \mathbb{R}^m$.
2. We represent our present decisions (here and now) by a variable $x \in \mathbb{R}^n$. This decision implies a cost $I(x)$.
3. After a realization ξ, we make another decision y_ξ at a cost $c(y_\xi, \xi)$. Thus our total cost is

$$F(x, \xi) = I(x) + c(y_\xi, \xi)$$

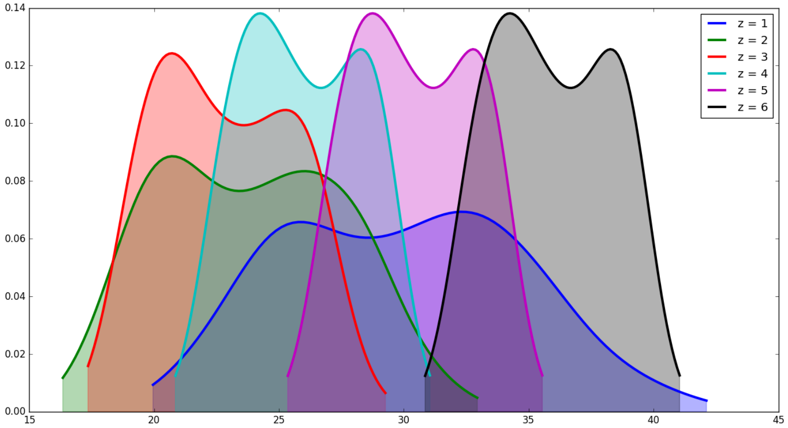
For each here and now decision x, our future (uncertain) cost is described by a random variable.

$$x \to G(x) := F(x, \cdot)$$

We need to choose among random variables.

Risk

$$x \to G(x) := F(x, \cdot)$$



Risk

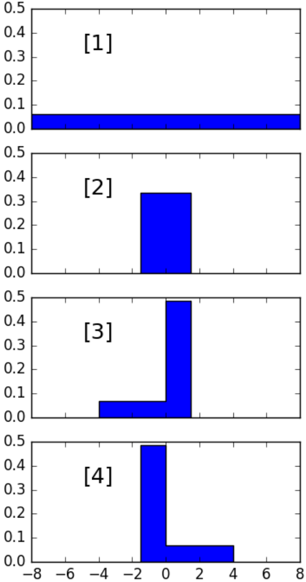
Is it enough to use the expected value for choosing among random variables?

$$\begin{aligned}\min_{x \in X} \mathbb{E}[G(x)] &= \min_{x \in X} \mathbb{E}[F(x,\xi)] \\ &= \min_{x \in X} I(x) + \mathbb{E}[c(y_\xi,\xi)] \\ &= \min_{x \in X} I(x) + \mathbb{E}[Q(x,\xi)]\end{aligned}$$

Is it enough to use the expected value for choosing among random variables?

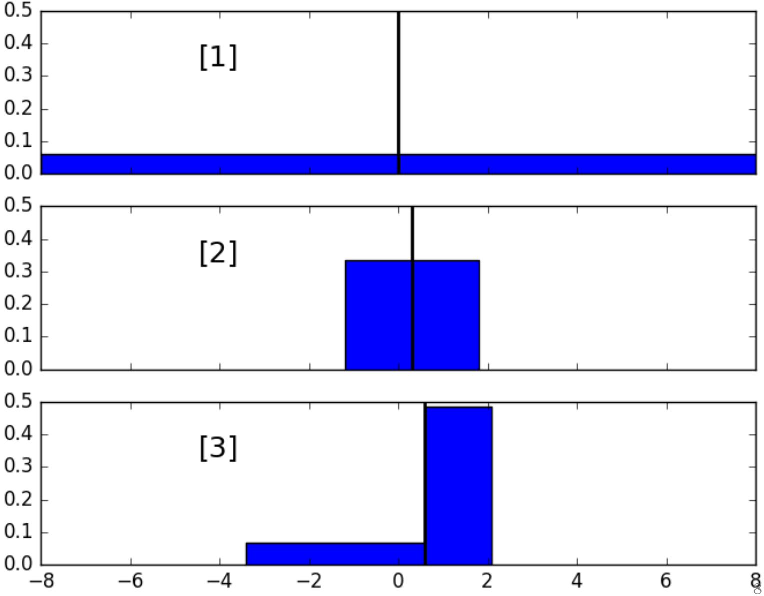
- After a realization of ξ, $F(x, \xi)$ may be very different from $\mathbb{E}[F(x, \xi)]$
- From an average cost (profit) point of view, using expected value, is reasonable as long as bad results does not lead you out of game.

Risk



1. Extreme (good and bad) results may happen with positive probability. High risk!!
2. High good outcomes are equally not likely as bad ones. Low risk.
3. Only bad high gad outcomes are not likely. Low risk.

Risk



"The concept of Risk is imprecise, it embodies the idea of the real chance of bad things to happen."

On the context of stochastic programming, we assume we have a random variable Z describing an uncertain quantity (cost, profit, etc) for which we want to measure the "risk" it embodies. A risk measure is function

$$\rho : \mathcal{M} \to \overline{\mathbb{R}}$$
$$Z \to \rho(Z)$$

where $\mathcal{M}$ is some space of random variables (e.g $L^p(\Omega), p \geq 1$)
Example: $\rho = \mathbb{E}$.

Quantile

Given a random variable Z with c.d.f. $F_Z : \mathbb{R} \to [0,1]$, with $F_Z(t) = \mathbb{P}[Z \leq t]$, we have that

- F_Z is increasing, right continuous.
- $F_Z(-\infty) = 0$ and $F_Z(\infty) = 1$.
- For any $0 < c < 1$ there exist $a \in \mathbb{R}$ such that

$$\{t : F_Z(t) \geq c\} = [a, +\infty\rangle$$

- **Definition.** For $0 < c < 1$, we define the c-quantile as $F^{-1}(c) := min\{t : F_Z(t) \geq c\}$.
- $F_Z(F_Z^{-1}(c)) \geq c$.
- If F_Z is continuous at $F_Z^{-1}(c)$, then $F_Z(F_Z^{-1}(c)) = c$.

Risk Measures
VaR

Given a random variable Z and $0 < \alpha < 1$, we define the value at risk of Z at level α by

$$VaR_\alpha(Z) := F_Z^{-1}(1-\alpha)$$

- $\mathbb{P}[Z > VaR_\alpha(Z)] \leq \alpha$. Worse outcomes that $VaR_\alpha(Z)$ occurs with probability less that α.
- For any $\varepsilon > 0$, $\mathbb{P}[Z > VaR_\alpha(Z) - \varepsilon] > \alpha$. "$VaR_\alpha(Z)$ is the smallest threshold that ensure bad outcomes happening with probability less that α."

Properties of Risk Measures

Translation Equivariance

A risk measure ρ is translation equivariant if

$$\rho(Z + a) = \rho(Z) + a, \quad \forall Z, \forall a \in \mathbb{R}$$

“ If a random variable is perturbed by a (deterministic) quantity, its risk should be affected by the same amount”
Example $\mathbb{E}, VaR_\alpha$.

Properties of Risk Measures

Positive Homogeneity

A risk measure ρ is positively homogeneous if

$$\rho(aZ) = a\rho(Z), \quad \forall Z,\ \forall a \geq 0$$

Example $\mathbb{E}, VaR_\alpha$.

Properties of Risk Measures

Monotonicity

A risk measure ρ is monotone if

$$\rho(Z) \leq \rho(Y), \quad \text{if } Z \leq Y, [a.e.]$$

" If random variable is systematically better than another, then it should be less risky"

Example $\mathbb{E}, VaR_\alpha$.

Properties of Risk Measures

Subadditivity

A risk measure ρ is subadditive if

$$\rho(Y+Z) \leq \rho(Y) + \rho(Z), \quad \forall Z, Y$$

" A merger does not increase risk"

Example $\mathbb{E}$.

VaR is not convex

Consider $\alpha = 0.1$ and two i.i.d. random variables X and Y whose distribution density function is

$$f(x) = 0.9\chi_{[-1,0]}(x) + 0.05\chi_{[0,2]}(x)$$