Two-stage Stochastic Programming Problems

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The Interchangeability Property

Two Stage Problem

$$
\text{First Stage}\begin{cases} \min_x & f(x) + \mathbb{E}[Q(x,\xi)] \\ \text{s. t.} & x \in X \end{cases}
$$

$$
\text{Second Stage}\begin{cases} Q(x,\xi) := & \min_y \quad g(y,\xi) \\ & \text{s. t.} \quad y \in \mathcal{G}(x,\xi) \end{cases}
$$

- It is crucial $Q(x,\cdot)$ to be measurable, for all x.
- Easy case: ξ discrete random vector (since $Q(x,\cdot)$ is a discrete random variable).
- Is the second stage minimizer function y_ξ measurable?.
- Example: Consider
  - $\Omega = [0,1]$ with the Lebesgue measure.
  - $\xi(\omega) = \omega$ (uniform distribution)
  - $\mathcal{G}(x,\xi) = [0,1]$
  - $g(y,\xi) = \text{sign}\big((\xi - y\chi_N(y))^2\big)$, where $N \subset [0,1]$ is a non mensurable set.

The Interchangeability Property

One Stage Problem

$$
\begin{array}{cl}
\min_{x,y_\xi} & f(x) + \mathbb{E}[g(y_\xi,\xi)] \\
\text{s. t.} & x \in X \\
 & y_\xi \in \mathcal{G}(x,\xi), \quad \text{a.e. } [\xi] \\
 & {\color{red} y_\xi \text{ is measurable}}
\end{array}
$$

- $g(y_\xi, \xi)$ needs to be measurable, for all x.

The Interchangeability Property

Proposition

Assuming that $Q(x, \xi)$ and $g(y_\xi, \xi)$ are measurable, then the one stage and two stage problems are equivalent.

The Interchangeability Property

Definition

A function $g : \mathbb{R}^m \times \Omega \to \overline{\mathbb{R}}$ is a Carathéodory function if

1. $g(y, \cdot)$ is measurable for all y.
2. $g(\cdot, \omega)$ is continuous for a. e. $[\omega]$.

Theorem

If g is a Carathéodory function and the probability measure (on Ω), then g is measurable over $\mathbb{R}^m \times \Omega$.

The Interchangeability Property

Given the problem

$$G(\omega) := \left.\begin{array}{ll} \min_y & g(y,\omega) \\ \text{s. t.} & c(y,\omega) \le 0 \end{array}\right\} (P)$$

Theorem

If all components of $c(y,\omega)$ are Carathéodory functions, then there exists a family $\{c^k\}_{k=1}^{\infty}$ of measurable functions such that for each ω, $\{c^k(\omega)\}_{k=1}^{\infty}$ is a dense subset of $\{y : c(y,\omega) \le 0\}$.

Theorem

If $g(y,\omega)$ and all components of $c(y,\omega)$ are Carathéodory functions, then $G(\omega)$ is measurable. Also, if $\mathbb{E}[G] \in \mathbb{R}$, and $\bar{y}(\omega)$ is a minimizer of (P), then $\bar{y}$ is measurable and solves

$$\begin{array}{lll} \mathbb{E}[G] = & \min_y & \mathbb{E}[g(y_\omega,\omega)] \\ & \text{s. t.} & c(y_\omega,\omega) \le 0 \\ & & y_\omega \text{ is measurable} \end{array}$$