TWO-STAGE SLP
EXPECTED RECOURSE FUNCTION

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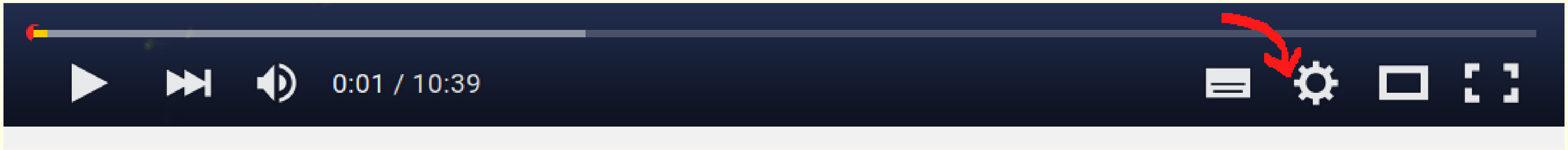
Set YouTube resolution to 480p



for best viewing

Questions on the homework (mandatory for PhD students)

In

- $h_1 = 180 + \zeta_1$ com $\zeta_1 \sim \mathcal{N}(0, 16)$.
- $h_2 = 163 + \zeta_2$ com $\zeta_2 \sim \mathcal{N}(0, 9)$.

the first parameter in the normal distribution $\mathcal{N}$ corresponds to the mean. And the 2nd one, is it the standard deviation (σ) or the variance (σ^2)?

The recourse function (fixed ξ)

Under reasonable assumptions

$$Q(x,\xi) = \left\{ \begin{array}{rl} \min & q^\top y \\ \text{s.t.} & Wy = h - Tx \\ & y \geq 0 \end{array} \right. = \left\{ \begin{array}{rl} \max & \pi^\top (h - Tx) \\ \text{s.t.} & \pi \in \Pi(q) \\ \Pi(q) & = \{\pi : W^\top \pi \leq q\} \end{array} \right.$$

We wrote $Q(x,\xi) = v(h - Tx)$ for suitable v and saw that

$$\partial v(z_0) = \arg\max \left\{ \pi^\top z_0 : \pi \in \Pi \right\}$$

The subdifferential of $Q(\cdot, \xi)$

Since $Q(x_0, \xi) = \nu(h - Tx_0)$, a chain rule gives

$$\partial \mathbf{Q}(\mathbf{x_0}, \xi) = -\mathbf{T}^\top \arg\max \left\{ \pi^\top (\mathbf{h} - \mathbf{T}\mathbf{x_0}) : \pi \in \Pi \right\}$$

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- ξ has a finite support: K scenarios

- ξ is a general distribution

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The subdifferential of $\mathbb{E}[Q(\cdot, \xi)]$

- ξ has a finite support: K scenarios

  easier case

- ξ is a general distribution

  less easy case: depends on recourse structure

The subdifferential of $\mathbb{E}[Q(\cdot,\xi)]$: ξ with finite support

K **scenarios**

$$\partial\mathbb{E}[\mathbf{Q}(\mathbf{x_0},\xi)] = -\sum_{\mathbf{k=1}}^{\mathbf{K}} \mathbf{p_k}\mathbf{T^k}^{\top} \arg\max\left\{\pi^{\top}(\mathbf{h^k}-\mathbf{T^k}\mathbf{x_0}) : \pi\in\Pi(\mathbf{q^k})\right\}$$

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First concern: when is the expected recourse function **well-defined**?

The subdifferential of $\mathbb{E}[Q(\cdot,\xi)]$: ξ with finite support

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The subdifferential of $\mathbb{E}[\mathbf{Q}(\cdot,\xi)]$: general ξ

First concern: when is the expected recourse function **well-defined**?

Answer: $Q(x,\cdot)$ needs to be **measurable** and either the expected surplus $\mathbb{E}[Q(x,\xi)_+]$ or the expected shortage $\mathbb{E}[Q(x,\xi)_-]$ must be finite (Props. 2.6, 2.7)

A super fast introduction to measurability

What is the size of

- Intervals $I_{[a,b]} = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, \ldots, n\}$?

- Closed bounded set $A \in \mathbb{R}^n$?

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- Any set in $\mathbb{R}^n$?

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  A covering C of A is $C = \cup_j I_{[a^j,b^j]}) \supset A$ with $I^{j_1} \cap I^{j_2} = \emptyset$

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- Closed unbounded set $A \in \mathbb{R}^n$?

  A is measurable if $A \cap I_{[a,b]}$ is measurable, for all intervals

- Any set in $\mathbb{R}^n$?

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$\mathcal{A}$ is a class of measurable sets A in $\mathbb{R}^n$ with measure μ if

- $A \in \mathcal{A}$ implies $\mathbb{R}^n \backslash A \in \mathcal{A}$
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Then $\cap_i A_i \in \mathcal{A}$, $\mu(A) \geq 0$, $\mu(\emptyset) = 0$, $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ if $A_i \cap A_j = \emptyset$

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and for an abstract space $\boxed{\Omega}$?

$\mathcal{F}$ is a class of measurable sets F in Ω with probability measure P

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A **random vector** $\xi : \Omega \to \mathbb{R}^n$ is such that for all $A \in \mathcal{A}$

$$\xi^{-1}[A] = \{\omega : \xi(\omega) \in A\} \in \mathcal{F}$$

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$$P_\xi(A) = P(\{\omega : \xi(\omega) \in A)\})$$

A simple probability space

$\Omega = \{\neq \text{ qualities of a box of bananas}\}$ $\quad \xi(\Omega) = \{0\} \cup \{1/2\} \cup \{[\![1,8]\!]\}$

1. unusable
2. good for cooking
3. good for eating raw
   a) appearance
   b) flavour
   c) fragrance

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for $a, t, f \in [0,1]$. The random variable ν helps pricing the bananas:

$$$ if $\nu(\omega) \in (6,8]$; $$ if $\nu(\omega) \in (4,6]$; $ if $\nu(\omega) \in (1,4]$

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$\Omega = \{\neq \text{qualities of a box of bananas}\}$ $\boxed{v(\Omega) = \Big\{\{0\} \cup \{1/2\} \cup \{[1,8]\}\Big\}}$

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   a) appearance
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for $a, t, f \in [0, 1]$. The random variable v helps pricing the bananas:

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Finite first moment

$Q(x, \cdot)$ needs to be **measurable**

either the expected surplus $\mathbb{E}[Q(x, \xi)_+]$ or the expected shortage $\mathbb{E}[Q(x, \xi)_-]$ must be finite

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- Fixed recourse
- Simple fixed recourse
- Complete fixed recourse
- Relatively Complete fixed recourse

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Let's study $\phi(x) := \mathbb{E}(Q(x,\xi)$ when recourse is simple and fixed

Illustration

Consider the SLP in $\mathbb{R}$ and its recourse problem

$$\begin{cases} \min & cx \\ \text{s.t.} & x = \omega \text{ a.e.} \\ & x \geq 0 \end{cases}$$

$$Q(x, \omega) = \begin{cases} \min & q^+ y^+ + q^- y^- \\ \text{s.t.} & y^+ + y^- = \omega - x \\ & y^+, y^- \geq 0 \end{cases}$$

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The 2SLP is

$$\begin{cases} \min & cx + \mathbb{E}[Q(x,\omega)] \\ \text{s.t.} & x \geq 0 \end{cases}$$

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The 2SLP is

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– $\Pi(q) \neq \emptyset \iff q^+ + q^-$ (sufficiently expensive recourse)

– when $q^+ = q^- = 1$, $Q(x,\omega) = |\omega - x|$

– Ω with finite support, $\mathbb{E}[Q(x,\omega)]$ is convex and polyhedral, with kinks $\omega \in \Omega$

– Ω continuous