Multistage stochatic programming problems

Optimality conditions

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NESTED FORMULATION

- $\xi = (\xi_1, \dots, \xi_T)$ is the stochastic process
- $f_t : \Re^{n_t} \times \Re^{d_t} \to \bar{\Re}$, $t = 1, \dots, T$, are continuous functions
- $x_t \in \Re^{n_t}$, $t = 1, \dots, T$, are the decision variables
- $\mathcal{X}_t : \Re^{n_{t-1}} \times \Re^{d_t} \rightrightarrows \Re^{n_t}$,$t = 1, \dots, T$, are measurable, closed valued multifunctions

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$$\inf_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}\left[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} f_2(x_2, \xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}[\inf_{x_T \in \mathcal{X}(x_{T-1}, \xi_T)} f_T(x_T, \xi_T)]\right]\right].$$

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$$\inf_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E}_{|\xi_1} \left[\inf_{x_2 \in \mathcal{X}_2(x_1, \xi_2)} f_2(x_2, \xi_2) + \mathbb{E}_{|\xi_{[2]}} \left[\cdots + \mathbb{E}_{|\xi_{[T-1]}} \left[\inf_{x_T \in \mathcal{X}(x_{T-1}, \xi_T)} f_T(x_T, \xi_T) \right] \right] \right].$$

DYNAMIC PROGRAMMING FORMULATION

- Stage $t = T$

$$Q_T(x_{T-1}, \xi_{[T]}) := \inf_{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)} f_T(x_T, \xi_T)$$

- At stages $t = 2, \ldots, T-1$

$$Q_t(x_{t-1}, \xi_{[t]}) := \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} f_t(x_t, \xi_t) + \mathbb{E}_{|\xi_{[t]}} \left[Q_{t+1}(x_t, \xi_{[t+1]}) \right]$$

- Stage $t = 1$

$$\inf_{x_1 \in \mathcal{X}_1} f_1(x_1) + \mathbb{E} \left[Q_2(x_1, \xi_2) \right]$$

RECOURSE FUNCTION

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) := \mathbb{E}_{|\xi_{[t]}} \left[Q_{t+1}(x_t, \xi_{[t+1]}) \right]$$

COST-TO-GO FUNCTION

$$Q_t(x_{t-1}, \xi_{[t]}) := \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$$

Dynamic programming formulation

If the stochastic process is stagewise independent the problem becomes less "heavy"

Recourse function

$$\mathcal{Q}_{t+1}(x_t) := \mathbb{E}\left[Q_{t+1}(x_t, \xi_{t+1})\right]$$

Cost-to-go function

$$Q_t(x_{t-1}, \xi_t) := \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t)$$

General formulation

Consider $x_t := x_t(\xi_{[t]})$, $t = 1, \ldots, T$ as functions of the stochastic process up to stage t: $\xi_{[t]}$.

Definition

The mapping

$$x_t : \Re^{d_1} \times \cdots \times \Re^{d_t} \to \Re^{n_t}$$

is called an *implementable policy*.

An implementable policy is said to be feasible if

$$x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t]}), \xi_t), \quad t = 2, 3, \ldots, T \quad w.p.\ 1.$$

General formulation

$$\begin{cases} \inf & \mathbb{E}\left[f_1(x_1) + f_2(x_2(\xi_{[2]}), \xi_2) + \cdots + f_T(x_T(\xi_{[T]}), \xi_T)\right] \\ \text{s.t} & x_1 \in \mathcal{X}_1 \\ & x_t(\xi_{[t]}) \in \mathcal{X}_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \quad t = 2, \dots, T \end{cases}$$

- Function $x_t(\xi_{[t]})$ is measurable with respect to the σ-algebra $\mathcal{F}_t$.

Unless the data process has finitely many scenarios, the above is an infinite dimensional optimization problem.

General formulation

$$\left\{\begin{array}{ll} \inf & \mathbb{E}\left[f_1(x_1) + f_2(x_2(\xi_2), \xi_2) + \cdots + f_T(x_T(\xi_T), \xi_T)\right] \\ \text{s.t} & x_1 \in \mathcal{X}_1 \\ & x_t(\xi_t) \in \mathcal{X}_t(x_{t-1}(\xi_{t-1}), \xi_t), \quad t = 2, \ldots, T \\ & x_t(\xi_t) \lhd \mathcal{F}_t, \qquad t = 1, \ldots, T \end{array}\right.$$

- $x_t(\xi_t) \lhd \mathcal{F}_t$ means that the function $x_t(\xi_t)$ is measurable with respect to the σ-algebra $\mathcal{F}_t$.

Unless the data process has finitely many scenarios, the above is an infinite dimensional optimization problem.

Assumptions

Feasible sets

- $X_1 := \{x_1 \in \Re^{n_1} : A_1 x_1 = b_1\}$
- $X_t(x_{t-1}, \xi_t) := \{x_t \in \Re^{n_t} : B_t x_{t-1} + A_t x_t = b_t\}$

Functions

- $f_t : \Re^{n_t} \times \Re^{d_t} \to \bar{\Re}$, $t = 2, \ldots, T$, are random lower semicontinuous functions (rlsc)
- $f_t(\cdot, \xi_t)$ is convex for a.e. ξ_t, and $t = 1, \ldots, T$

Definition. We say that $f(x, \xi)$ is a rlsc function if the associated epigraphical multifunction $\xi \to epi(f(x, \xi))$ is closed and measurable.

- If there exist constraints of the type $x_t \geq 0$, we'll assume that

$$f_t(x_t, \xi_t) = \infty \quad \text{if} \quad x_t \ngeq 0\,.$$

Assumptions

We implicitly assume that the data (A_t, B_t, b_t) depends on the uncertainties

$$(A_t, B_t, b_t) = (A_t(\xi_t), B_t(\xi_t), b_t(\xi_t)), \quad t = 1, \dots, T$$

We'll focus on the dynamic programming formulation

Recourse function

$$\mathcal{Q}_{t+1}(x_t, \xi_{[t]}) := \mathbb{E}_{|\xi_{[t]}}\left[Q_{t+1}(x_t, \xi_{[t+1]})\right]$$

Cost-to-go function

$$Q_t(x_{t-1}, \xi_{[t]}) := \inf_{x_t}\{f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) : B_t x_{t-1} + A_t x_t = b_t\}$$

Convexity

Lemma

Let $\psi_t(x_t, \xi_{[t]}) := f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$ *be a convex function of* x_t. *Then the value function*

$$v(y) := \inf_{x_t} \{\psi_t(x_t, \xi_{[t]}) : A_t x_t = y\}$$

is convex.

Convexity

Lemma

Given the assumptions on f_t, the problem

$$\inf_{x_1}\{f_1(x_1) + \mathcal{Q}_2(x_1, \xi_{[1]}) : A_1 x_1 = b_1\}$$

is convex.

Optimality conditions

Additional assumption (★)

For all small perturbations of the vector b_t, the corresponding optimal value

$$Q_t(x_{t-1}, \xi_{[t]}) := \inf_{x_t} \{ f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) : B_t x_{t-1} + A_t x_t = b_t \}$$

is finite

Associated to the above problem we define the Lagrangian function

$$L_t(x_t, \pi_t) = f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) + \pi_t^\top (b_t - B_t x_{t-1} - A_t x_t)$$

We denote the set of dual solution by

$$\mathcal{D}_t(x_t, \xi_{[t]}) := \arg\sup_{\pi_t} \inf_{x_t} L_t(x_t, \pi_t)$$

Optimality conditions

Propositon

Let $\psi_t(x_t, \xi_{[t]}) := f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]})$. *If* (★) *holds, then*

i) *there is no duality gap:*

$$\begin{aligned} Q_t(x_{t-1}, \xi_{[t]}) \quad &= \sup_{\pi_t} \inf_{x_t} L_t(x_t, \pi_t) \\ &= \sup_{\pi_t} \left\{ -\psi_t^*(A_t^\top \pi_t, \xi_{[t]}) + \pi_t^\top (b_t - B_t x_{t-1}) \right\} \end{aligned}$$

Optimality conditions

Propositon

Let $\psi_t(x_t,\xi_{[t]}) := f_t(x_t,\xi_t) + \mathcal{Q}_{t+1}(x_t,\xi_{[t]})$. *If* ($\bigstar$) *holds, then*

i) *there is no duality gap:*

$$\begin{aligned} Q_t(x_{t-1},\xi_{[t]}) &= \sup_{\pi_t} \inf_{x_t} L_t(x_t,\pi_t) \\ &= \sup_{\pi_t} \left\{ -\psi_t^*(A_t^\top \pi_t, \xi_{[t]}) + \pi_t^\top (b_t - B_t x_{t-1}) \right\} \end{aligned}$$

ii) $\bar{x}_t$ *is a primal optimal solution iff there exists* $\bar{\pi}_t = \bar{\pi}_t(\xi_{[t]})$ *such that* $\bar{\pi}_t \in \mathcal{D}_t(x_{t-1},\xi_{[t]})$ *and* $0 \in \partial_x L_t(\bar{x}_t,\bar{\pi}_t)$

iii) *The function* $Q_t(x_{t-1},\xi_{[t]})$ *is subdifferentiable at* x_{t-1} *and* $\partial_x Q_t(x_{t-1},\xi_{[t]}) = -B_t^\top \mathcal{D}_t(x_{t-1},\xi_{[t]})$.

Optimality conditions

The optimality conditions of the dynamic equations can be written as

$$\bar{x}_t(\xi_{[t]}) \in \arg\min_{x_t}\{f_t(x_t,\xi_t) + \mathcal{Q}_{t+1}(x_t,\xi_{[t]}) : \; A_t x_t = b_t - B_t x_{t-1}\}$$

Since the objective function is convex and the constraints are linear, a feasible policy is optimal iff it satisfies the following optimality conditions.

Optimality condition

For all $t = 1,\ldots,T$ and a.e. $\xi_{[t]}$ there exists $\bar{\pi}_t(\xi_{[t]})$ such that the following conditions holds:

$$0 \in \partial[f_t(\bar{x}_t(\xi_{[t]}),\xi_t) + \mathcal{Q}_{t+1}(\bar{x}_t(\xi_{[t]}),\xi_{[t]})] - A_t^\top \bar{\pi}_t(\xi_{[t]})\,.$$

In order to obtain a more practical expression we will need the following assumption.

Assumption ($\square$)

For all $t = 2,\ldots,T$ and a.e. $\xi_{[t]}$ the function $\mathcal{Q}_t(\cdot,\xi_{[t-1]})$ is finite valued.

Optimality conditions

Propositon

Suppose that assumptions (★) and (□) are satisfied. A feasible policy $\bar{x}_t(\xi_{[t]})$ is optimal iff there exists mappings $\bar{\pi}_t(\xi_{[t]})$, $t = 1, \dots, T$, such that the condition

$$0 \in \partial f_t(\bar{x}_t(\xi_{[t]}), \xi_t) - A_t^\top \bar{\pi}_t(\xi_{[t]}) + \mathbb{E}_{|\xi_{[t]}}[\partial Q_{t+1}(\bar{x}_t(\xi_{[t]}), \xi_{[t+1]})]$$

holds true for a.e. $\xi_{[t]}$ and $t = 1, \dots, T$.
Moreover, multipliers $\bar{\pi}_t(\xi_{[t]})$ satisfies the above inclusion iff for a.e. $\xi_{[t]}$ it holds that

$$\bar{\pi}_t(\xi_{[t]}) \in \mathcal{D}_t(\bar{x}_{t-1}(\xi_{[t-1]}), \xi_{[t]}) \, .$$

Optimality conditions

Theorem

Suppose that assumptions (★) and (□) are satisfied. A feasible policy $\bar{x}_t(\xi_{[t]})$ is optimal iff there exists measurable $\bar{\pi}_t(\xi_{[t]})$, $t = 1, \ldots, T$, such that

$$0 \in \partial f_t(\bar{x}_t(\xi_{[t]}), \xi_t) - A_t^\top \bar{\pi}_t(\xi_{[t]}) - \mathbb{E}_{|\xi_{[t]}}[B_{t+1}^\top \bar{\pi}_{t+1}(\xi_{[t+1]})$$

for a.e. $\xi_{[t]}$ and $t = 1, \ldots, T$.

Optimality conditions

The previous theorem (and also the two previous propositions) hold true if Assumptions ($\bigstar$) and ($\square$) are replaced by the following one

Assumption ($*$)

The functions $f_t(x_t, \xi_t)$, $t = 1, \ldots, T$, are random polyhedral and the number of scenarios is finite.

Definition

Function $g(x, \omega)$ is called random polyhedral if g can be written as

$$g(x,\omega) = \begin{cases} \max_{j \in J} \gamma(\omega) + q_j(\omega)^\top x & \text{if} \quad d_k(\omega)^\top x \leq r_k(\omega) \;\; \forall\, k \in K \\ \infty & \text{otherwise,} \end{cases}$$

where J and K are finite index sets.